Extreme Analysis of a Random Ordinary Differential Equation

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November 19, 2013

Abstract

In this paper, we consider a one dimensional stochastic system described by an elliptic equation. A spatially varying random coefficient is introduced to account for uncertainty or imprecise measurements. We model the logarithm of this coefficient by a Gaussian process and provide asymptotic approximations of the tail probabilities of the derivative of the solution.

1 Introduction

In this paper, we consider the tail event that arises naturally from a differential equation employed in various applications. Very often, microscopic heterogeneity or uncertainty of parameters exists such that the system cannot be completely characterized by a deterministic differential equation. Stochastic models are usually employed, in combination with differential equations, to account for such heterogeneity and/or uncertainty. In this paper, we are interested in one specific differential equation concerning a real-valued solution $v(x)$

$$(a(x)v'(x))' = p(x), \quad x \in [0, L]. \tag{1}$$

where $a(x)$ and $p(x)$ are real-valued functions. This equation has applications to several subfields of physics and also has a close connection to stochastic differential equations.

In this paper, we adopt the formulation that the process $a(x)$ is a spatially varying stochastic process and thus the corresponding solution $v(x)$ itself (as a function of $a(x)$) is also a stochastic process. In physical models, the process $a(x)$ is constrained to be positive. A natural modeling approach is that $a(x)$ is a log-normal process, that is,

$$a(x) = e^{-\sigma\xi(x)}, \quad \sigma > 0 \tag{2}$$

where $\xi(x)$ is a Gaussian process living on $[0, L]$. We are interested in developing sharp asymptotic approximations of the tail probabilities associated with $v(x)$, in particular,

$$w(b) \triangleq P(\max_x |v'(x)| > b) \quad \text{as } b \to \infty. \tag{3}$$

Such tail probabilities serve as a risk measure of elastic material failure based on the maximum strain (*i.e.* $v'(x)$) criteria [11].

Under the Dirichlet boundary condition, $u(0) = u(L) = 0$, and with representation (2), equation (1) has a closed form solution $v(x) = \int_0^x F(t)e^{\sigma\xi(t)}dt - \int_0^x e^{\sigma\xi(t)}dt \times \int_0^L F(s)e^{\sigma\xi(s)}ds / \int_0^L e^{\sigma\xi(s)}ds$

*This research is supported in part by NSF CMMI-1069064 and NSF SES-1123698.

where $F(x) \triangleq \int_0^x p(t)dt$ and its derivative is

$$v'(x) = e^{\sigma\xi(x)} \left\{ F(x) - \frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt} \right\}. \quad (4)$$

The contribution of this paper is the derivation of a closed form sharp asymptotic approximations of $w(b)$ as $b \to \infty$. In particular, we discuss two situations: $p(x)$ is a constant and $|p(x)|$ admits one unique maximum in the interior of $[0, L]$. In addition to the asymptotic approximations of $w(b)$, this analysis also implies qualitative descriptions of the most likely sample path along which $\max_x |v'(x)|$ achieves a high level. First, if $p(x)$ is a constant, then the maximum of $|v'(x)|$ is likely to be obtained at either end of the interval and it is unlikely to be obtained in the interior. Second, if $|p(x)|$ admits one unique interior maximum at $x_* = \arg\max_x |p(x)|$, then the maximum of $|v'(x)|$ is likely to be obtained at either of the three locations, 0, L, or close to x_*, depending on the specific values of $p(0)$, $p(L)$, and $p(x_*)$.

Upon considering $\max |v'(x)|$ as a functional of the input Gaussian process $\xi(x)$, the current analysis sits well in the literature of rare-event analysis for Gaussian processes. An incomplete of literature includes [7, 15, 1, 13, 4, 14, 5, 8, 9, 10, 2, 3, 12]. The analysis combines physics understanding, which helps with guessing the most probable sample path of $\xi(x)$ given the high excursion of $|v'(x)|$, and random field techniques to derive approximations of $w(b)$.

The rest of this paper is organized as follows. In Section 2, we present the main results. The theorems are proved in Sections 3. A supplementary material is provided at http://stat.columbia.edu/~jcliu/paper/OneDimDirichletDensity26FinalSupplement.pdf. A more comprehensive manuscript is available at http://arxiv.org/abs/1309.3851 containing more discussions on the applications.

2 Main results

We consider the differential equation (1) with the Dirichlet condition. The gradient of the solution is given by (4). The random coefficient $a(x)$ takes the form (2), where $\xi(x)$ is a Gaussian process living on $[0, L]$. We list a set of technical conditions concerning the input process $\xi(x)$ and $p(x)$.

A1 The process $\xi(x)$ is strongly stationary and furthermore $E[\xi(x)] = 0$ and $E[\xi^2(x)] = 1$.

A2 The process $\xi(x)$ is almost surely three-time differentiable. The covariance function admits the following expansion $Cov(\xi(0), \xi(x)) = C(x) = 1 - \frac{\Delta}{2}x^2 + \frac{A}{24}x^4 - Bx^6 + o(x^6)$, as $x \to 0$. In addition, for each x, $C(\lambda x)$ is a non-increasing function of $\lambda \in \mathbb{R}^+$.

A3 The function $p(x)$ is at least twice continuously differentiable. In addition, it falls into either of the two cases.

Case 1. $|p(x)|$ admits its unique interior global maximum $x_* = \arg\max |p(x)|$ and $x_* \in (0, L)$. Furthermore, $|p(x)|$ is strongly concave (meaning that the second derivative is strictly negative) in a sufficiently small neighborhood around x_*.

Case 2. $p(x)$ is constant.

Assumption A2 is an important assumption for the entire analysis. In particular, three-time differentiability implies that the covariance function is at least six-time differentiable and the first, the third, and the fifth derivatives evaluated at the origin are all zero. The coefficients Δ and A are known as the spectral moments that will be further discussed in the later analysis. In Assumption A3, if $|p(x)|$ has more than one (interior) global maxima or the global maximum is at the boundary, the analysis can be adapted.

In the following, we first consider Case 1 that $|p(x)|$ admits one unique maximum. Let $x_* \triangleq \arg\max_{x\in[0,L]} |p(x)|$ be the unique interior maximum in $(0, L)$. Without loss of generality, we assume that $p(x_*)$, $p(0)$, and $p(L)$ are all positive. For the case that some or all of them are negative, the analysis is completely analogous. This will be mentioned in later remarks.

We define three variables u, u_0 and u_L that depend on the excursion level b. They are all approximately on the scale of $\frac{\log b}{\sigma}$. For each $b > 0$, let u be the solution to the nonlinear equation

$$p(x_*)H(\gamma_*(u), u)e^{\sigma u} = b, \tag{5}$$

where

$$H(x, u) \triangleq |x|e^{-\frac{1}{2}\Delta\sigma u x^2} \tag{6}$$

and $\gamma_*(u) \triangleq \arg\sup_{x>0} H(x, u) = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. Identity (5) can be simplified to $\frac{p(x_*)}{\sqrt{\sigma\Delta u}}e^{\sigma u-\frac{1}{2}} = b$. We introduce the notation $\gamma_*(u)$ and H because they arise naturally in the derivation and have geometric and probabilistic interpretations that will be given in the proof of our main theorems.

For each $b > 0$, let u_0 be the solution to $\frac{e^{\sigma u_0}}{\sqrt{\Delta\sigma u_0}} \times \sup_{\{(x,\zeta):x\leq\zeta\}} H_0(x, \zeta; u_0) = b$ where

$$H_0(x, \zeta; u) \triangleq e^{-\frac{x^2}{2}} \times E\Big[p(0)(x - Z) + \frac{p'(0)}{2\sqrt{\Delta\sigma u}}(x - Z)^2 \ \Big|\ Z \leq \zeta\Big]. \tag{7}$$

Z is a standard Gaussian random variable independent of any other randomness in the system; $E[\cdot|Z \leq \zeta]$ denotes the conditional expectation with respect to Z given $Z \leq \zeta$. We provide further explanations of H_0. The second term inside the expectation (7) is $o(1)$ and thus $H_0(x, \zeta; u) \approx p(0)e^{-x^2/2}(x - E[Z|Z \leq \zeta])$. The last term in the definition of H_0 is important to obtain a sharp approximation of the tail probabilities. More properties of H_0 are included in Remark 1. Similarly, we define u_L by

$$\frac{e^{\sigma u_L}}{\sqrt{\Delta\sigma u_L}} \times \sup_{\{(x,\zeta):x\leq\zeta\}} H_L(x, \zeta; u_L) = b. \tag{8}$$

where $H_L(x, \zeta; u)$ is defined similar as in (7) by replacing $p(0)$ and $p'(0)$ by $p(L)$ and $-p'(L)$, respectively.

Function $F(x)$ is bounded and the factor, $F(x) - \int_0^L F(t)e^{\xi(t)}dt / \int_0^L e^{\xi(t)}dt$, is also bounded. In fact, this factor converges to zero under the conditional distribution given the high excursion of $|v'(x)|$. Thus, if $|v'(x)|$ exhibits a high excursion, then $\xi(x)$ must also achieve a high level. The variable u is interpreted as the level which $\xi(x)$ needs to achieve so that $|v'(x)|$ achieves the level b around x_*. Similarly, u_0 and u_L correspond to the high excursion levels of $\xi(x)$ at the two ends.

For each ζ, u_0, and u_L, maximizing $\log(|H_0|)$ and $\log(|H_L|)$ over $x \in (-\infty, \zeta]$ gives the definitions of the following functions: $G_0(\zeta; u_0) \triangleq \sup_{x\leq\zeta} \log |H_0(x, \zeta; u_0)|$ and $G_L(\zeta; u_L) \triangleq \sup_{x\leq\zeta} \log |H_L(x, \zeta; u_L)|$. Define the maximizers of the G-function $\zeta_0 \triangleq \arg\max_\zeta G_0(\zeta; u_0)$, and $\zeta_L \triangleq \arg\max_\zeta G_L(\zeta; u_L)$. Note that ζ_0 depends on u_0 and ζ_L depends on u_L. To simplify the notation, we omit the indices u_0 and u_L in the notation ζ_0 and ζ_L when there is no ambiguity. The second derivatives of the G-functions evaluated at their maximizers are $\Xi_0 \triangleq -\lim_{u_0\to\infty} \partial_\zeta^2 G_0|_{\zeta=\zeta_0,u=u_0}$, and $\Xi_L \triangleq -\lim_{u_L\to\infty} \partial_\zeta^2 G_L|_{\zeta=\zeta_L,u=u_L}$, respectively. Lastly, we define constant

$$\kappa_0 \triangleq \frac{A\zeta_0}{24\Delta^2\sigma} - \frac{A \times E\left[Z^4|Z \leq \zeta_0\right]}{24\Delta^2\sigma} + \frac{E[\frac{p''(0)}{6\sigma\Delta}(\zeta_0 - Z)^3 + \frac{Ap(0)}{24\Delta^2\sigma^2}Z^4(\zeta_0 - Z)\,|Z \leq \zeta_0]}{p(0)E[\zeta_0 - Z\,|Z \leq \zeta_0]}, \tag{9}$$

as well as κ_L which is similar to the above by replacing ζ_0 with ζ_L. The main results are summarized in the following theorems.

Theorem 1 *Suppose that $\xi(x)$ is a Gaussian process satisfying conditions A1 - A2 and Case 1 of A3. For all $x \in [0, L]$, let $v'(x)$ be given as in (4). Let u, u_0, and u_L be defined above. If $p(x)$ is nonnegative at $x = 0$, x_*, and L, then $P(\sup_{x\in[0,L]} |v'(x)| > b) \sim D \times u^{-1/2}e^{-u^2/2} + D_0 \times u_0^{-1}e^{-u_0^2/2} + D_L \times u_L^{-1}e^{-u_L^2/2}$ where D, D_0, and D_L are constants defined as*

$$D = \frac{\sqrt{\Delta}e^{\frac{A}{24\sigma^2\Delta^2}+\frac{p''(x_*)}{6p(x_*)\sigma^2\Delta}}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \int \exp\left\{-\frac{1}{2}\left[\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} - \frac{y^2 z}{\Delta} + \frac{Ay^4}{4\Delta^4} + \frac{Ay^2}{2\sigma\Delta^3} - \frac{p''(x_*)y^2}{p(x_*)\sigma\Delta^2}\right]\right\} dydz.$$

$$D_0 = \frac{\sqrt{\Delta}e^{\kappa_0/\sigma}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \int e^{-\frac{1}{2}\left(\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} + \frac{\Xi_0}{\Delta}y^2\right)} dydz, \quad D_L = \frac{\sqrt{\Delta}e^{\kappa_L/\sigma}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \int e^{-\frac{1}{2}\left(\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} + \frac{\Xi_L}{\Delta}y^2\right)} dydz.$$

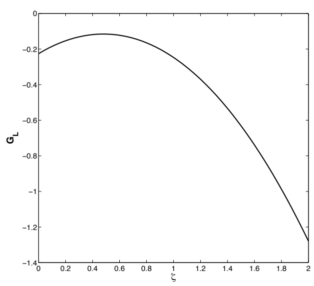

Figure 1: Function $G_L(\zeta, u_L = \infty)$.

If $p(x)$ attains its maximum at multiple interior points $x_1, \cdots, x_k$, then the approximation becomes $P(\sup_{x\in[0,L]} |v'(x)| > b) \sim \sum_{j=1}^k D(j)u^{-1/2}e^{-u^2/2} + D_0 u_0^{-1}e^{-u_0^2/2} + D_L u_L^{-1}e^{-u_L^2/2}$, where $D(j)$'s are defined similarly as D by replacing x_* with x_k. If the maximizer x_* is attained on the boundary, then the term $Du^{-1/2}e^{-u^2/2}$ should be removed from the approximation.

The theorem assumes that $p(x)$ is positive at the important locations. In the case when $p(x_*) < 0$, we simply define u through $|p(x_*)|e^{\sigma u + H(\gamma_*(u),u)} = b$. The definitions of other variables remain. Similarly, if $p(0)$ is negative we should generally define that $H_0(x, \zeta; u) \triangleq \mathrm{sign}(p(0))e^{-\frac{x^2}{2}} \times E\Big[p(0)(x - Z) + \frac{p'(0)}{2\sqrt{\Delta\sigma u}}(x - Z)^2 \Bigm| Z \le \zeta\Big]$, where "sign" is the sign function. The same treatment can be applied to H_L when $p(L)$ is negative. The rest of the definitions remains. To simplify the notation, we assume that $p(0)$ and $p(L)$ are positive and do not include the sign term.

Remark 1 *There are several features of the functions H_0 and H_L that are important in the analysis. As $u_L \to \infty$, we have that $H_L(x, \zeta; u_L) \to p(L)e^{-x^2/2}(x - E[Z|Z \le \zeta]) > 0$ and $\zeta_L \approx 0.48$. In addition, for $\zeta \le 0.84$, we have $\frac{\partial|H_L|}{\partial x}|_{(x,\zeta)=(\zeta,\zeta)} > 0$, and thus $\max_{x\in(-\infty,\zeta]} \log|H_L(x, \zeta)|$ is solved at $x = \zeta$, that is, $G_L(\zeta; u_L) = \log|H_L(\zeta, \zeta; u_L)|$. This calculation is important in the technical derivations and it ensures that the maximum of $|v'(x)|$ is attained precisely at $x = L$ if $\max_{L-\varepsilon<x\le L} |v'(x)| > b$. To assist understanding, we numerically computed the function G_L for $\zeta > 0$ by setting $u_L = \infty$ and plot it in Figure 1 for $p(L) = 1$.*

Now, we proceed to the approximation of $w(b)$ when $p(x) \equiv p_0 > 0$. The approximation is very similar to Theorem 1, except that we do not have the term $D \times u^{-1/2}e^{-u^2/2}$ and all the derivatives of $p(x)$ vanish. To state the theorem, we need the following notation. We define a similar H-function and G-function as $H_h(x, \zeta) = p_0 e^{-\frac{x^2}{2}} E[x - Z|Z \le \zeta]$, and $G_h(\zeta) = \sup_{x\le\zeta} \log|H_h(x, \zeta)|$.

Furthermore, we define constants $\zeta_h = \arg\sup_\zeta G_h(\zeta)$, $\Xi_h = -\partial_\zeta^2 G_h(\zeta_h)$,

$$D_h = \frac{\Delta e^{\kappa_h/\sigma}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \times \int \exp\Big\{-\frac{1}{2}\Big[\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} + \frac{\Xi_h}{\Delta} y^2\Big]\Big\} dydz$$

$$\kappa_h = \frac{A\zeta_h^4}{24\Delta^2\sigma} - \frac{AE\left[Z^4|Z\le\zeta_h\right]}{24\Delta^2\sigma} + \frac{AE[Z^4(\zeta_h - Z)\,|Z\le\zeta_h]}{24\Delta^2\sigma^2 E[\zeta_h - Z\,|Z\le\zeta_h]}.$$

Theorem 2 *Suppose that the random field $\xi(x)$ satisfies the Conditions A1-A2 and case 2 of A3. In addition, the external force $p(x)\equiv p_0$ is a positive constant. For each $b>0$, let u_h solve $\frac{e^{\sigma u_h}}{\sqrt{\Delta\sigma u_h}} \times \sup_{\{(x,\zeta):x\le\zeta\}} H_h(x,\zeta) = b$. Then, we have the closed form approximation $P(\sup_{x\in[0,L]}|v'(x)|>b) \sim 2D_h u_h^{-1} e^{-u_h^2/2}$.*

The proof of Theorem 2 is very similar to that of Theorem 1. We present it in the supplementary material. We further provide intuitive interpretations of the previous asymptotic approximations. In particular, we focus mostly on the case when $p(x)$ is not a constant.

The approximation in Theorem 1 consists of three pieces. The first term $Du^{-1/2}e^{-u^2/2}$ corresponds to the probability that the maximum of $|v'(x)|$ is attained close to the interior point $x_* = \arg\max_{x\in[0,L]}|p(x)|$; the terms $D_0 u_0^{-1} e^{-u_0^2/2}$ and $D_L u_L^{-1} e^{-u_L^2/2}$ correspond to the probabilities that the excursion of $|v'(x)|$ occurs at the two boundary points $x=0$ and $x=L$, respectively. Thus, this three-term decomposition of $w(b)$ suggests that the conditional probability $P(\max_{x\in[\varepsilon,x_*-\varepsilon]\cup[x_*+\varepsilon,L-\varepsilon]}|v'(x)|>b \mid \max_{[0,L]}|v'(x)|>b) \to 0$ as $b\to\infty$ for any $\varepsilon>0$. It is unlikely that the maximum is attained at a location other than the two ends or x_*. As for which of the three locations is most likely to exhibit a high excursion, it depends on the specific functional forms of $p(x)$. Note that all the three terms decay exponentially fast with u^2, u_0^2, or u_L^2. Therefore, the smallest among u, u_0, and u_L corresponds to the most likely location. Note that u_0 and u_L take the same form. Thus, we only need to compare $|p(0)|$ and $|p(L)|$. The larger one corresponds to a smaller u-value and therefore yields a more likely high excursion. To compare the boundary case and the interior case, we need to compare u and u_0 (or u_L). We take u_0 as an example. Note that both u and u_0 are defined by b implicitly through the equations in similar forms. Therefore, it is sufficient to compare among the two terms

$$|p(x_*)e^{H(\gamma_*,u)}| = |p(x_*)|\frac{e^{-1/2}}{\sqrt{\sigma\Delta u}} \text{ and } \frac{\sup_{x\le\zeta} H_0(x,\zeta,u_0)}{\sqrt{\sigma\Delta u_0}} \sim \frac{|p(0)|\sup_{x\le\zeta} e^{-x^2/2}E[x-Z|Z\le\zeta]}{\sqrt{\sigma\Delta u}}.$$

Furthermore, we consider the ratio

$$r \triangleq \frac{\sup_{x\le\zeta} e^{-x^2/2}E[x-Z|Z\le\zeta]}{\sqrt{\sigma\Delta u}} \Big/ \frac{e^{-1/2}}{\sqrt{\sigma\Delta u}} = \sup_{(\zeta,x),s.t.\ x\le\zeta} e^{\frac{1-x^2}{2}} E[x-Z|Z\le\zeta].$$

Note that r is a universal constant strictly greater than 1. If $|p(x_*)| > r|p(0)|$, then x_* is a more probable location to observe a high excursion; if $|p(x_*)| < r|p(0)|$, then zero is a more probable location. If $p(x)$ is a constant, then $u > u_0 = u_h$. This is why the maximum of $v'(x)$ is not attained in the interior for this case.

3 Proof of Theorem 1

To make the discussion smooth, we present the proof of all supporting propositions and lemmas in the supplementary material. The proof in Theorem 1 is based on the following inclusion-exclusion formula $\sum_{i=1}^3 P(E_i) - \sum_{i=1}^2 \sum_{j=i+1}^3 P(E_i \cap E_j) \le P(\max_{[0,L]} v'(x) > b) = P(\cup_{i=1}^3 E_i) \le \sum_{i=1}^3 P(E_i)$, where $E_1 = \{\max_{x\in[u^{-1/2+\delta}, L-u^{-1/2+\delta}]} |v'(x)| \ > b\}$, $E_2 = \{\max_{x\in[0,u^{-1/2+\delta}]} |v'(x)| \ > b\}$, and $E_3 = \{\max_{x\in[L-u^{-1/2+\delta},L]} |v'(x)| \ > b\}$, for some $\delta > 0$ sufficiently small but independent of b. The main body is to derive the approximations for $P(E_i)$. In addition, from the following detailed derivation of $P(E_1)$ and $P(E_3)$, it is straight forward to have that

$$P(E_1 \cap E_2) + P(E_1 \cap E_3) + P(E_2 \cap E_3) = o(P(E_1) + P(E_2) + P(E_3)). \tag{10}$$

Thus, we complete the proof of Theorem 1 by the inclusion-exclusion formula. In the following analysis, we use both x and t to denote the spatial index. In particular, we use t for the index when doing integration and use x when taking the supremum.

3.1 Approximation for $P(E_1)$

Consider the following change of variables from $(\xi(x_*), \xi'(x_*), \xi''(x_*))$ to (w, y, z) that depends on the variable u, $w \triangleq \xi(x_*) - u$, $y \triangleq \xi'(x_*)$, and $z \triangleq u + \xi''(x_*)/\Delta$. We further write $P(\cdot|\xi(x_*) = u + w, \xi'(x_*) = y, \xi''(x_*) = -\Delta(u-z)) = P(\cdot|w,y,z)$ and obtain

$$P(E_1) = \Delta \int P(E_1|w,y,z)h(w,y,z)dwdydz. \tag{11}$$

where $h(w,y,z)$ is the density function of $(\xi(x_*), \xi'(x_*), \xi''(x_*))$ evaluated at $(u+w, y, -\Delta(u-z))$. The following proposition localizes the event to a region convenient for Taylor expansion on $\xi(x)$.

Proposition 1 *Under the conditions in Theorem 1, consider* $\mathcal{L}_u = \{|w| < u^{3\delta}\} \cap \{|y| < u^{1/2+4\delta}\} \cap \{|z| < u^{1/2+4\delta}\}$. *Then, for any* $\delta > 0$, *we have that* $P(\mathcal{L}_u^c; E_1) = o(u^{-1}e^{-u^2/2})$.

This proposition localizes the event E_1 to a region where the maximum of $v'(x)$ is achieved around x_*. The above proposition suggests that we only need to consider the event on the set $\mathcal{L}_u$, that is, $\Delta \int_{\mathcal{L}_u} P(E_1|w,y,z)h(w,y,z)dwdydz$.

Conditional on $(\xi(x_*), \xi'(x_*), \xi''(x_*))$, we write the process in the following representation $\xi(x) = E(\xi(x)|w,y,z) + g(x-x_*)$. The process $g(x-x_*)$ represents the variation of $\xi(x)$ when $\xi(x_*)$ and its first two derivatives have been fixed. Thus, $g(x-x_*)$ is a mean-zero Gaussian process almost surely three-time differentiable. Using conditional Gaussian calculations and Taylor expansion, we have that $Var(g(x-x_*)) = O(|x-x_*|^6)$, that is, $g(x-x_*) = O_p(|x-x_*|^3)$ as g is the remainder term after conditioning on $\xi(x_*)$ and the first two derivatives. Note that the distribution of $g(x)$ is free of (w,y,z). Let $\bar{E}(x;w,y,z) \triangleq E(\xi(x)|w,y,z)$. By means of the conditional Gaussian calculations (Chapter 5.5 [4]), we have that $\partial \bar{E}(x_*;w,y,z) = y$, $\partial^2 \bar{E}(x_*;w,y,z) = -\Delta(u-z)$, $\partial^3 \bar{E}(x_*;w,y,z) = -\frac{A}{\Delta}y$, and $\partial^4 \bar{E}(x_*;w,y,z) = Au + O(z)$, where "$\partial$" is the partial derivative with respect to x. We perform Taylor expansion on $\bar{E}(x;w,y,z)$. Using the notation $\vartheta(x) = O(u^{1/2+4\delta}x^4 + ux^6)$, we obtain that on the set $\mathcal{L}_u$

$$\begin{aligned}
\xi(x) =& u + w + y(x-x_*) - \frac{\Delta(u-z)}{2}(x-x_*)^2 \\
&- \frac{A}{6\Delta}y(x-x_*)^3 + \frac{Au}{24}(x-x_*)^4 + g(x-x_*) + \vartheta(x-x_*) \\
=& u + w + \frac{y^2}{2\Delta(u-z)} - \frac{\Delta(u-z)}{2}\Big(x - x_* - \frac{y}{\Delta(u-z)}\Big)^2 \\
&- \frac{A}{6\Delta}y(x-x_*)^3 + \frac{Au}{24}(x-x_*)^4 + g(x-x_*) + \vartheta(x-x_*).
\end{aligned} \tag{12}$$

For $\delta > 0$, we further localize the event by the following proposition.

Proposition 2 *For each $\delta, \delta' > 0$ chosen small enough and $\delta' > 24\delta$, we have that*

$$P\Big(\sup_{|x|>u^{-1/2+8\delta}}(|g(x)| - \delta' u x^2) > 0 \quad or \quad \sup_{|x|\le u^{-1/2+8\delta}} |g(x)| > u^{-1/2+\delta'},\ \mathcal{L}_u\Big) \quad = \quad o(u^{-1}e^{-u^2/2}).$$

With this proposition, let $\mathcal{L}'_u = \mathcal{L}_u \cap \{\sup_{|x|>u^{-1/2+8\delta}}[|g(x)| - \delta' u x^2] < 0\} \cap \{\sup_{|x|\le u^{-1/2+8\delta}} |g(x)| < u^{-1/2+\delta'}\}$. We further reduce the event to $\Delta \int_{\mathcal{L}_u} P(E_1, \mathcal{L}'_u | w, y, z) h(w, y, z) dwdydz$.

Step 1: $v'(x)$**.** It is necessary to be reminded that the derivations are on the set $\mathcal{L}'_u$. Consider the change of variable that $s = s(x) : x \to \sqrt{\Delta(u-z)}\Big(x - x_* - \frac{y}{\Delta(u-z)}\Big)$. We insert s to the expansion in (12) and obtain that

$$\begin{aligned} \xi(x) \quad = \quad & u + w + \frac{y^2}{2\Delta(u-z)} - \frac{Ay^4}{8\Delta^4(u-z)^3} - \frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}} s \\ & - \frac{Ay^2}{4\Delta^3(u-z)^2} s^2 + \frac{A}{24\Delta^2(u-z)} s^4 + g(x - x_*) + \vartheta(x - x_*) + o(s^4 u^{-5/4}). \end{aligned} \tag{13}$$

To begin with, we are interested in approximating

$$F(x) - \frac{\int_0^L F(t) e^{\sigma\xi(t)} dt}{\int_0^L e^{\sigma\xi(t)} dt} = \frac{\int_0^L (F(x) - F(t)) e^{\sigma\xi(t)} dt}{\int_0^L e^{\sigma\xi(t)} dt}. \tag{14}$$

To compute the integration, it is convenient to write the terms in the above expansion formula for $\xi(x)$ that do not include x (or equivalently s) as $c_* \triangleq \sigma\left[u + w + \frac{y^2}{2\Delta(u-z)} - \frac{Ay^4}{8\Delta^4(u-z)^3}\right]$. We first consider the denominator

$$\begin{aligned} \int_0^L e^{\sigma\xi(x)} dx \quad = \quad & e^{c_*} \int_0^L \exp\Big\{\sigma\Big[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}} s - \frac{Ay^2}{4\Delta^3(u-z)^2} s^2 \\ & + \frac{A}{24\Delta^2(u-z)} s^4 + g(x - x_*) + \vartheta(x - x_*)\Big]\Big\} dx, \end{aligned}$$

and separate it into two parts

$$\int_0^L e^{\sigma\xi(x)} dx \quad = \quad \int_{|x-x_*|<u^{-1/2+8\delta}} e^{\sigma\xi(x)} dx + \int_{|x-x_*|\ge u^{-1/2+8\delta}} e^{\sigma\xi(x)} dx = J_1 + J_2. \tag{15}$$

According to Assumption A2 and on the set $\{\sup_{|x|>u^{-1/2+8\delta}}[|g(x)| - \delta' u x^2] \le 0\}$ (δ' can be chosen arbitrarily small), there exists some $\varepsilon_0 > 0$ so that the minor term

$$J_2 = \int_{|x-x_*|\ge u^{-1/2+8\delta}} e^{\sigma\xi(x)} dx \le \int_{|x-x_*|\ge u^{-1/2+8\delta}} e^{c_* - 2\varepsilon_0 u(x-x_*)^2} \le e^{c_* - \varepsilon_0 u^{16\delta}}.$$

We now proceed to the dominating term J_1. Note that, on the set $|x - x_*| < u^{-1/2+8\delta}$, $\vartheta(x - x_*) = o(u^{-1})$. Then, we obtain that

$$J_1 = \frac{e^{c_* + o(u^{-1})}}{\sqrt{\Delta(u-z)}} e^{\omega(u)} \times$$
$$\int_{|x-x_*|<u^{-1/2+8\delta}} \exp\left\{\sigma\left[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}} s - \frac{Ay^2}{4\Delta^3(u-z)^2} s^2 + \frac{A}{24\Delta^2(u-z)} s^4\right]\right\} ds,$$

where $\omega(u) = O(\sup_{|x|\le u^{-1/2+8\delta}} |g(x)|)$. Since $Var(g(x)) = O(|x|^6)$, it is helpful to keep in mind that $\omega(u) = O_p(u^{-3/2+24\delta})$.

Lemma 1 *On the set $\mathcal{L}'_u$, we have that*

$$\int_{|x-x_*|<u^{-1/2+8\delta}} e^{\sigma[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}} s - \frac{Ay^2}{4\Delta^3(u-z)^2} s^2 + \frac{A}{24\Delta^2(u-z)} s^4]} ds = \sqrt{\frac{2\pi}{\sigma}} e^{-\frac{Ay^2}{4\Delta^3(u-z)^2} + \frac{A}{8\Delta^2\sigma u} + o(u^{-1})}.$$

We insert the result of the above lemma into the expression of J_1 term, put J_1 and J_2 terms together, and obtain that on the set $\mathcal{L}'_u$

$$\int_0^L e^{\sigma\xi(x)} dx = \sqrt{\frac{2\pi}{\sigma\Delta(u-z)}} \exp\left\{c_* - \frac{Ay^2}{4\Delta^3(u-z)^2} + \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\}. \qquad (16)$$

We now proceed to the analysis of (14). Let $\tau_* = x_* + \gamma_*$, where $\gamma_* = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. For each $x - \tau_* = O(u^{-1/2+16\delta})$, we define change of variable $\gamma = x - x_* - \frac{y}{\Delta(u-z)}$. Note that $\xi(x)$ is approximately a quadratic function with maximum at $x_* + \frac{y}{\Delta(u-z)}$. Thus, γ is approximately the distance to the mode of $\xi(x)$. Similar to the derivations of Lemma 1 and using the results in (16), the following lemma provides an approximation of (14).

Lemma 2 *On the set $\mathcal{L}'_u$, we have that*

$$\begin{aligned} F(x) - \frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt} &= p(x)\gamma \exp\Big\{ - \frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\ &\qquad + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} + o(u^{-1}) + \omega(u)\Big\}. \end{aligned} \qquad (17)$$

We apply the change of variable $\gamma = x - x_* - \frac{y}{\Delta(u-z)}$ to the representation of $\xi(x)$ in (12) and obtain that

$$\begin{aligned} \xi(x) &= u + w + \frac{y^2}{2\Delta(u-z)} - \frac{\Delta(u-z)}{2}\gamma^2 - \frac{A}{6\Delta} y(\gamma + \frac{y}{\Delta(u-z)})^3 + \frac{Au}{24}(\gamma + \frac{y}{\Delta(u-z)})^4 \\ &\quad + g(x - x_*) + \vartheta(x - x_*). \end{aligned} \qquad (18)$$

We now put together (17) and (18) and obtain that for $|x - x_*| \le u^{-1/2+8\delta}$

$$\begin{aligned} v'(x) &= e^{\sigma u + \sigma w + \frac{\sigma y^2}{2\Delta(u-z)}} \times p(x)\gamma \times e^{-\frac{\sigma\Delta u}{2}\gamma^2} \qquad (19) \\ &\quad \times \exp\Big\{\frac{\sigma\Delta z}{2}\gamma^2 - \frac{\sigma A}{6\Delta} y(\gamma + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24}(\gamma + \frac{y}{\Delta(u-z)})^4 \\ &\qquad - \frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} + o(u^{-1}) + \omega(u)\Big\}. \end{aligned}$$

Step 2: the event $E_1 = \{\max_{x\in[u^{-1/2+\delta}, L-u^{-1/2+\delta}]} |v'(x)| > b\}$**.** By the definition of u and the analytic form of (19), we have that $v'(x) \geq b = p(x_*)\gamma_* e^{\sigma u - \frac{\Delta\sigma u}{2}\gamma_*^2}$ if and only if $\gamma > 0$ and

$$\begin{aligned}
&\sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{\sigma\Delta z}{2}\gamma^2 - \frac{\sigma A}{6\Delta} y(\gamma + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24}(\gamma + \frac{y}{\Delta(u-z)})^4 \\
&- \frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\
&+ \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} + \log H(\gamma, u) - \log H(\gamma_*, u) + \log\frac{p(x)}{p(x_*)} \quad \geq o(u^{-1}) - \omega(u),
\end{aligned} \tag{20}$$

where H is defined as in (6) and $\gamma_* = \frac{1}{\sqrt{\sigma\Delta u}}$. We write the left-hand side of the above display as $R(\gamma) + \log H(\gamma, u) - \log H(\gamma_*, u)$. Note that $\partial_\gamma^2 \log H(\gamma_*, u) = -2\Delta\sigma u$ and the derivative of the remainder term is $\partial_\gamma R(\gamma_*) = o(1) + O(z\gamma_*)$. Thus, $\log H(\gamma, u)$ dominates the variation. In particular, the left-hand side of (20) is maximized at $\gamma = \gamma_* + o(u^{-1}) + O(z\gamma_*/u) = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2} + o(u^{-1}) + O(z\gamma_*/u)$, equivalently, at $x = x_* + \gamma_* + y/\Delta(u-z) + o(u^{-1}) + O(z\gamma_*/u)$. Therefore, $\max_{|\gamma|\leq u^{-1/2+8\delta}} R(\gamma) + \log H(\gamma, u) - \log H(\gamma_*, u) = R(\gamma_*) + o(u^{-1}) + O(z^2/u^2)$. This is interpreted as $\max_{|x-x_*|\leq u^{-1/2+8\delta}} v'(x) \geq b$ if and only if

$$\begin{aligned}
\mathcal{A} \triangleq &\sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{\sigma\Delta z}{2}\gamma_*^2 - \frac{\sigma A}{6\Delta} y(\gamma_* + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24}(\gamma_* + \frac{y}{\Delta(u-z)})^4 \\
&- \frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) \\
&+ \frac{Ay^3}{3\Delta^4(u-z)^3\gamma_*} + \log\frac{p(x_* + \gamma_* + \Delta^{-1}(u-z)^{-1}y)}{p(x_*)} + O(z^2/u^2) \quad \geq \quad o(u^{-1}) - \omega(u).
\end{aligned} \tag{21}$$

Note that on the region $|x - x_*| > u^{-1/2+8\delta}$ we need to consider the variation of $g(x - x_*)$. On the set $\mathcal{L}'_u$, the variation of $v'(x)$ is dominated by $\log H(\gamma, u)$. In particular, on the set $|x - x_*| > u^{-1/2+8\delta}$, we have $\log H(\gamma, u) - \log H(\gamma_*, u) \leq -\varepsilon_0 u(\gamma - \gamma_*)^2$. Furthermore, on the set $\mathcal{L}'_u$, we have that $\sup_{|x|>u^{-1/2+8\delta}}(|g(x)| - \delta' u x^2) < 0$. We can choose $\delta' < \varepsilon_0/2$, then $2|g(x)| < \log H(\gamma_*, u) - \log H(\gamma, u)$ for all $|x - x_*| > u^{-1/2+8\delta}$. Thus, on the set $\mathcal{L}'_u$, the maximum of $v'(x)$ is attained on $|x - x_*| \leq u^{-1/2+8\delta}$, i.e. $\max_{[u^{-1/2+\delta}, L-u^{-1/2+\delta}]} v'(x) > b$ if and only if $\mathcal{A} > o(u^{-1}) - \omega(u)$. The following lemma simplifies the analytic form of $\mathcal{A}$.

Lemma 3 *The expression $\mathcal{A}$ can be simplified to* $\mathcal{A} = \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2} y^2 z + \frac{z}{2u} + \frac{A}{24\sigma\Delta^2 u} + \frac{p''(x_*)}{6p(x_*)\sigma\Delta u} - \frac{\sigma A y^4}{8\Delta u^3} + \frac{y^2}{u^2}(-\frac{A}{4\Delta^3} + \frac{p''(x_*)}{2p(x_*)\Delta^2}) + o(u^{-1} + y^2 u^{-2}) + O(z^2/u^2)$.

With exactly the same development, we have $\max_{x\in[u^{-1/2+\delta}, L-u^{-1/2+\delta}]}[-v'(x)] \geq b$ if and only if $\mathcal{A} \geq o(u^{-1}) + \omega(u)$. In fact, from the technical proof of Lemma 3, we basically choose $\gamma = -\gamma_* + o(u^{-1}) + O(z\gamma_*/u)$ and all the other derivations are the same. We omit the repetitive details. Thus, the event E_1 occurs if and only if $\mathcal{A} \geq o(u^{-1}) + \omega(u)$.

Step 3: evaluation of the integral in (11).

Lemma 4 *The density of $(\xi(x), \xi''(x), \xi''(x))$ evaluated at $(u + w, y, -\Delta(u - z))$ is* $h(w, y, z) = \frac{e^{-S(w,y,z)/2}}{(2\pi)^{3/2}\sqrt{\Delta(A-\Delta^2)}}$ *where* $S(w, y, z) = u^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + 2u(w + \frac{y^2}{2\Delta u})$.

The proof of the above lemma is elementary and therefore is omitted; see also Chapter 5.5 in [4]. We insert the expression of $\mathcal{A}$ in Lemma 3 to the exponent of the density function

$$S(w,y,z) = u^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + 2u\Big[\frac{\mathcal{A}}{\sigma} - \frac{y^2 z}{2\Delta u^2} - \frac{z}{2\sigma u} - \frac{A}{24\sigma^2\Delta^2 u} - \frac{p''(x_*)}{6p(x_*)\sigma^2\Delta u} + \frac{Ay^4}{8\Delta^4 u^3} - \frac{y^2}{u^2}\Big(-\frac{A}{4\sigma\Delta^3} + \frac{p''(x_*)}{2p(x_*)\sigma\Delta^2}\Big) + o(u^{-1} + y^2u^{-2}) + O(z^2/u^2)\Big]. \quad (22)$$

The following lemma provides a lower bound of $S(w,y,z)$ for the dominated convergence theorem.

Lemma 5 *On the set* $\mathcal{L}'_u$

$$S(w,y,z) \geq u^2 + 2u\mathcal{A}/\sigma + \frac{\Delta^2}{A}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big)^2 + \frac{1+o(1)}{\sigma}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big) - \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\frac{y^2}{u} + O(1).$$

It is useful to keep in mind that $p''(x_*) < 0$. Let $\mathcal{A}_u = u\mathcal{A}$. Note that for each fixed $(\mathcal{A}_u, y, z)$, $w \to 0$ as $u \to \infty$. Furthermore, notice that $\omega(u) = O(\sup_{|x|\leq u^{-1/2+8\delta}} |g(x)|) = O_p(u^{-3/2+24\delta})$. We consider a change of variable from (w,y,z) to $(\mathcal{A}_u, y, z)$. The dominated convergence theorem and (22) yield that

$$\begin{aligned}
& \Delta \int_{\mathcal{L}_u} P\left(E_1, \mathcal{L}'_u | w,y,z\right) h(w,y,z) dwdydz \\
= & \frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \times \int_{\mathcal{L}_u} P\left(\mathcal{A} > \omega(u), \mathcal{L}'_u |\, w,y,z\right) e^{-\frac{1}{2}S(w,y,z)} dwdydz \\
\sim & \frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} \times \int_{\mathcal{L}_u} I\left(\mathcal{A}_u > 0\right) e^{-\frac{1}{2}S(w,y,z)} \frac{d\mathcal{A}_u}{\sigma u} dydz
\end{aligned}$$

For the last step, we use the fact that $P(\mathcal{L}'_u|w,y,z) \to 1$ and $P(\mathcal{A} > \omega(u), \mathcal{L}'_u|w,y,z) \to I(\mathcal{A}_u > 0)$ as $u \to \infty$. We insert the expression $S(w,y,z)$ as in (22) and set $w = 0$ (by the dominated convergence theorem and the fact that for fixed $\mathcal{A}_u$, y, and z, we have $w \to 0$ as $u \to \infty$). The above the display is

$$\begin{aligned}
\sim & \frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} u^{-1} e^{-u^2/2 + \frac{A}{24\sigma^2\Delta^2} + \frac{p''(x_*)}{6p(x_*)\sigma^2\Delta}} \times \int_0^\infty \frac{1}{\sigma} e^{-\mathcal{A}_u/\sigma} d\mathcal{A}_u \\
& \times \int \exp\left(-\frac{1}{2}\left[\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} - \frac{y^2 z}{\Delta u} + \frac{A}{4\Delta^4}\frac{y^4}{u^2} - \frac{y^2}{u}\left(-\frac{A}{2\sigma\Delta^3} + \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\right)\right]\right) dydz.
\end{aligned}$$

We apply the change of variable that $y_u = u^{-1/2}y$ for the integration, then the above display is

$$\sim \quad D \times u^{-1/2} e^{-u^2/2}. \quad (23)$$

This corresponds to the first term of the approximation in the statement of the theorem.

3.2 The approximation of $P(E_3)$

The analysis of $P(E_2)$ and $P(E_3)$ are analogous. We only need to derive $P(E_3)$. The difference between the analyses of $P(E_3)$ and $P(E_1)$ is that the integrals in the factor (14) are truncated by the boundary and therefore most of the calculations are related to conditional Gaussian distributions. We redefine some notation. Let u_L and ζ_L be defined as in Section 2 prior to the statement of the theorem. We first define $t_L = L - \frac{\zeta_L}{\sqrt{\Delta\sigma u_L}}$ that is the location where $\xi(x)$ is likely to have a high excursion given that $v'(x)$ has a high excursion at the right boundary L. We will perform Taylor expansion by conditioning on the field at t_L. We redefine notation (w, y, z) as $\xi(t_L) = u_L + w$, $\xi'(t_L) = y$, and $\xi''(t_L) = -\Delta(u_L - z)$. Furthermore, we consider the following change of variables "γ" and "s"

$$x = \gamma + t_L + \frac{y}{\Delta(u_L - z)}, \quad t = t_L + \frac{y}{\Delta(u_L - z)} + \frac{s}{\sqrt{\Delta(u_L - z)}}. \tag{24}$$

With simple calculations, we have that $t \le L$ if and only if $s \le \sqrt{\frac{(1-z/u_L)}{\sigma}}\zeta_L - \frac{y}{\sqrt{\Delta(u_L-z)}}$. Furthermore, it is useful to keep in mind that $v'(x)$ is maximized when γ is of order $u_L^{-1/2}$. Let $g(x)$ be the remainder process such that $\xi(x) = E(\xi(x)|w, y, z) + g(x - t_L)$. Similar to the analysis of $P(E_1)$, we first localize the event via the following proposition.

Proposition 3 *Using the notations in Theorem 1, under conditions A1 - A2, consider*

$$\begin{aligned} \mathcal{C}_{u_L} &= \{|w| > u_L^{3\delta}\} \cup \{|y| > u_L^{1/2+4\delta}\} \cup \{|z| > u_L^{1/2+4\delta}\} \\ &\quad \cup \Big\{ \sup_{|x|>u_L^{-1/2+8\delta}} [|g(x)| - \delta' u_L x^2] > 0 \Big\} \cup \Big\{ \sup_{|x|\le u_L^{-1/2+8\delta}} |g(x)| > u_L^{-1/2+\delta'} \Big\} \end{aligned}$$

Then, for any $\delta > 0$ and $\delta' > 24\delta$, we have that $P(\mathcal{C}_{u_L}; E_3) = o(u_L^{-1}e^{-u_L^2/2})$.

Let $\mathcal{L}^*_{u_L} = \mathcal{C}^c_{u_L}$ and we only need to consider $P(\mathcal{L}^*_{u_L}, E_3)$. With a similar derivation as that for $P(E_1)$, the following lemma provides an estimate of $\int_0^L (F(x) - F(t))e^{\sigma\xi(t)}dt / \int_0^L e^{\sigma\xi(t)}dt$.

Lemma 6 *On the set $\mathcal{L}^*_{u_L}$, we have that*

$$\begin{aligned} \frac{\int_0^L (F(x)-F(t))e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt} &= \frac{1}{\sqrt{\Delta\sigma u_L}} \times \exp\Big(\frac{z}{2u_L} - \frac{A}{24\Delta^2\sigma u_L}E\left[Z^4 | Z \le \zeta_L\right] + \lambda(u_L) + \omega(u_L)\Big) \times \qquad (25) \\ \Big\{ E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) &- \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big| \ Z \le \sqrt{1 - \tfrac{z}{u_L}}\zeta_L - \sqrt{\tfrac{\sigma}{\Delta(u_L-z)}}y \Big] \\ + E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 &+ \frac{Ap(x)}{24\Delta^2\sigma^2 u_L} Z^4 (\gamma\sqrt{\sigma\Delta u_L} - Z) \ \Big| \ Z \le \sqrt{1 - \tfrac{z}{u_L}}\zeta_L - \sqrt{\tfrac{\sigma}{\Delta(u_L-z)}}y \Big] \Big\} \end{aligned}$$

where $\lambda(u_L) = O(y^3/u_L^{5/2} + y^2/u_L^2 + y/u^{3/2}) + o(u_L^{-1} + u_L^{-1}z)$, $\omega(u) = O(\sup_{|x|\le u^{-1/2+8\delta}} |g(x)|)$, and Z is a standard Gaussian random variable.

Inside the "{ }" of the above approximation, the first expectation term is the dominating term and the second term is of order $O(u^{-1})$. The next lemma presents an approximation of $v'(x)$.

Lemma 7 *On the set $\mathcal{L}^*_{u_L}$, we have that*

$$\begin{aligned} v'(x) =& \exp\left(\lambda(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2}) + \omega(u_L) + \sigma u_L + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{A\sigma u_L}{24}\gamma^4\right) \\ & \times \frac{1}{\sqrt{\Delta\sigma u_L}} \exp\left(\frac{z}{2u_L} - \frac{A}{24\Delta^2\sigma u_L} E[Z^4|Z \le \zeta_L]\right) \\ & \times H_{L,x}\left(\gamma\sqrt{\sigma\Delta(u_L - z)}, \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y; u_L\right) \\ & \times \exp\left\{\frac{E\left[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L} Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z)\,|Z \le \zeta_L\right]}{p(x)E[\gamma\sqrt{\sigma\Delta u_L} - Z|Z \le \zeta_L]}\right\}, \end{aligned} \tag{26}$$

where $H_{L,y}(x, \zeta; u) \triangleq e^{-\frac{x^2}{2}} \times E\Big[p(y)(x - Z) - \frac{p'(y)}{2\sqrt{\Delta\sigma u}}(x - Z)^2 \;\Big|\; Z \le \zeta\Big]$.

Note that the definition of $H_{L,y}(x, \zeta; u)$ is slightly different from $H_L(x, \zeta, u)$ defined as in Section 2. In particular, if we let $y = L$, then $H_{L,y}(x, \zeta; u) = H_L(x, \zeta; u)$. Furthermore, according to the change of variable in (24), $x \le L$ if and only if

$$\gamma\sqrt{\sigma\Delta(u_L - z)} \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y. \tag{27}$$

Thus, the maximization of $v'(x)$ (in choosing the variable γ) is subject to the above constraint. According the definition of u_L in (8) and the notation $G_L(\zeta; u_L) = \sup_{x\le\zeta} \log|H_L(x, \zeta, u_L)|$, we have that $\max_{x\in[L-u^{-1/2+\delta}, L]} |v'(x)| > b$ if and only if

$$\begin{aligned} & \max_{x\in[L-u_L^{-1/2+\delta}, L]} \lambda(u_L) + \omega(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2}) \\ & + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{A\sigma u_L}{24}\gamma^4 + \frac{z}{2u_L} - \frac{AE\left[Z^4|Z \le \zeta_L\right]}{24\Delta^2\sigma u_L} \\ & + \log\left|H_{L,x}\left(\gamma\sqrt{\sigma\Delta(u_L - z)}, \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y;\ u_L\right)\right| - G_L(\zeta_L; u_L) \\ & + \frac{E[\frac{p''}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 + \frac{Ap}{24\Delta^2\sigma^2 u_L} Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z) \;\mid\; Z \le \zeta_L]}{p(x)E[\gamma\sqrt{\sigma\Delta u_L} - Z|Z \le \zeta_L]} \quad > 0. \end{aligned} \tag{28}$$

We now proceed to the evaluation of $P(E_3)$ that consists of two cases.

We first consider the case that $|\sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y - \zeta_L| \le \varepsilon$. Note that the major variation of the left-hand-side of (28) is dominated by

$$\log\left|H_{L,x}\left(\gamma\sqrt{\sigma\Delta(u_L - z)}, \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y; u_L\right)\right|. \tag{29}$$

Thanks to the discussion in Remark 1, the above expression is maximized at (subject to the constraint (27)) $\gamma\sqrt{\sigma\Delta(u_L - z)} = \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y$, that is,

$$\gamma = \frac{\zeta_L}{\sqrt{\Delta\sigma u_L}} - \frac{y}{\Delta(u_L - z)}. \tag{30}$$

Recall the change of variable in (24), this corresponds to $x = L$. That is, the maximum is attained on the boundary $x = L$. Then, we can replace $H_{L,x}$ in (28) by $H_{L,L} = H_L$. Let $\gamma_L = \frac{\zeta_L}{\sqrt{\sigma\Delta u_L}}$. For the particular choice of γ in (30), we have that $\gamma^4 = \gamma_L^4 + o(y^2/u_L^2)$. We have that $\max_{x\in[L-u_L^{-1/2+\delta},L]} |v_L'(x)| > b$ if and only if $\mathcal{A} \geq \omega(u_L)$ where

$$\begin{aligned}\mathcal{A} \triangleq & \lambda(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2}) + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{A\sigma u_L}{24}\gamma_L^4 + \frac{z}{2u_L} - \frac{AE\left[Z^4|Z\leq\zeta_L\right]}{24\Delta^2\sigma u_L} \\ & + G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y; u_L\Big) - G_L(\zeta_L;u_L) \\ & + \frac{E[\frac{p''(L)}{6\sigma\Delta u_L}(\gamma_L\sqrt{\sigma\Delta u_L}-Z)^3 + \frac{Ap(L)}{24\Delta^2\sigma^2 u_L}Z^4(\gamma_L\sqrt{\sigma\Delta u_L}-Z)\,|Z\leq\zeta_L]}{p(L)E[\zeta_L - Z|Z\leq\zeta_L]}.\end{aligned} \tag{31}$$

Lemma 8 *The expression $\mathcal{A}$ can be simplified to*

$$\mathcal{A} = \lambda(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2}) + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{z}{2u_L} + \frac{\kappa_L}{u_L} - \frac{\Xi_L + o(1)}{2}\Big(\frac{\zeta_L z}{2u_L} + \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2,$$

where κ_L is given as in (9).

With the above lemma, we rewrite $S(w,y,z)$ as

$$\begin{aligned} S(w,y,z) \quad = \quad & u_L^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + o(1) + o(y^2) \\ & + 2u_L\Big[\mathcal{A}/\sigma - \frac{z}{2\sigma u_L} - \frac{\kappa_L}{\sigma u_L} + \frac{\Xi_L + o(1)}{2\sigma}\Big(\frac{\zeta_L z}{2u_L} + \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2 \\ & \quad + \lambda(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2})\Big]. \end{aligned}$$

Similar to the derivation of (23), by the dominated convergence theorem, we have that

$$\begin{aligned} & P\left(\max_{x\in[L-u_L^{-1/2+\delta},L]} |v'(x)| > b\ ;\ \mathcal{L}^*_{u_L};\quad \left|\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y - \zeta_L\right| \leq \varepsilon\right) \\ \sim & \frac{\sqrt{\Delta}}{(2\pi)^{3/2}\sqrt{A-\Delta^2}} u_L^{-1} e^{-u_L^2/2 + \frac{\kappa_L}{\sigma}} \times \int \exp\left(-\frac{1}{2}\left(\frac{\Delta^2 z^2}{A-\Delta^2} - \frac{z}{\sigma} + \frac{\Xi_L}{\Delta}y^2\right)\right) dydz \\ = & D_L \times u_L^{-1} \times e^{-u_L^2/2}. \end{aligned} \tag{32}$$

The following lemma presents the case that $\left|\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y - \zeta_L\right| \geq \varepsilon$.

Lemma 9 *Under the conditions in Theorem 1, we have that*

$$P\left(\max_{x\in[L-u_L^{-1/2+\delta},L]} |v'(x)| > b; \mathcal{L}^*_{u_L};\quad \left|\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y - \zeta_L\right| \geq \varepsilon\right) = o(1)u_L^{-1}e^{-u_L^2/2}.$$

Combining (32), Lemma 9, and the localization result in Proposition 3, we have that

$$P\Big(\max_{x\in[L-u_L^{-1/2+\delta},L]} |v'(x)| > b\Big) \sim D_L \times u_L^{-1} e^{-u_L^2/2}.$$

Approximation of $P(E_2)$. The analysis of $P(E_2)$ is completely analogous. In particular, we let $t_0 = \frac{\zeta_0}{\sqrt{\Delta\sigma u_0}}$, $\xi(t_0) = u_0 + w$, $\xi'(t_0) = y$, and $\xi''(t_0) = -\Delta(u - z)$ and further adopt change of variables $x = t_0 + \frac{y}{\Delta(u_0 - z)} - \gamma$ and $t = t_0 + \frac{y}{\Delta(u_0-z)} - \frac{s}{\sqrt{\Delta(u_0-z)}}$. Then the calculations are exactly the same as those of $P(E_3)$. Therefore, we omit the repetitive derivations and provide the result that $P(\max_{x\in[0,u_L^{-1/2+\delta}]} |v'(x)| > b) \sim D_0 \times u_0^{-1} \times e^{-u_0^2/2}$. With the inclusion-exclusion formula and (10), we conclude the proof.

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Supplementary Material

A Proof of Theorem 2

Similar to the proof of Theorem 1, we consider the event E_1, E_2, and E_3 separately. By homogeneity and symmetry, $P(E_2) = P(E_3)$. The approximations of $P(E_2)$ and $P(E_3)$ are identical to those obtained in Section 3.2 by setting $p(x) \equiv p_0$. Therefore,

$$P(E_2) = P(E_3) \sim D_h u_h^{-1} e^{-u_h^{-2}/2}.$$

From the derivation of $P(E_2)$ in the previous proof, we obtain that $P(E_2 \cap E_3) = o(P(E_2))$. For the rest of the proof, we show that $P(E_1) = o(P(E_2))$ and thus $P(E_1 \cap E_2) = o(P(E_2))$.

Approximation of $P(E_1)$. Let $H(x, u)$ be as defined for Theorem 1 and u solve

$$p_0 H(\gamma_*(u), u) e^{\sigma u} = b,$$

where $\gamma_*(u) = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. For the rest of the proof, we will show that

$$P(E_1) = O(1)e^{-\frac{u^2}{2}+O(u^\varepsilon)}. \tag{33}$$

for any $\varepsilon > 0$. According to the discussion in Section 2, there exists an $\varepsilon_0 > 0$ such that $u > u_h + \varepsilon_0$ and thus $e^{-\frac{u^2}{2}+O(u^\varepsilon)} = o(1)u_h^{-1}e^{-u_h^{-2}/2}$. If the above bound in (33) can be established, then we can conclude the proof.

First, we derive an approximation for

$$\alpha(u, \varepsilon) = P\Big(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon}, \frac{L}{2}+u^{-1/2+\varepsilon}]} |v'(x)| > b\Big),$$

where $\varepsilon > 0$ is chosen small enough. Then, we split the region $[0, L]$ into $N = \frac{L}{2u^{-1/2+\varepsilon}}$ many intervals each of which is a location shift of $[0, 2u^{-1/2+\varepsilon}]$, i.e. $[2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon} + 2u^{-1/2+\varepsilon}]$. Thanks to the homogeneity of $\xi(x)$, the approximations for

$$P\left(\max_{x\in[2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon}+2u^{-1/2+\varepsilon}]} |v'(x)| > b\right)$$

are the same for all $1 \le k \le N-2$. Then, we have

$$P\left(\cup_{k=1}^{N-2}\{\max_{x\in[2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon}+2u^{-1/2+\varepsilon}]} |v'(x)| > b\}\right) \le (1+o(1))\frac{L}{2u^{-1/2+\varepsilon}}\alpha(u, \varepsilon).$$

In what follows, we derive an approximation for $\alpha(u, \varepsilon)$. The derivation is similar to the proof of the Theorem 1. Therefore, we omit the details and only lay out the key steps and the major differences. We expand $\xi(x)$ around $x = \frac{L}{2}$ conditional on (by redefining the notations)

$$\xi(\frac{L}{2}) = u + w, \quad \xi'(\frac{L}{2}) = y, \quad \xi''(\frac{L}{2}) = -\Delta(u - z)$$

and obtain that

$$\begin{aligned}\xi(x) &= u+w+\frac{y^2}{2\Delta(u-z)}-\frac{\Delta(u-z)}{2}\left(x-\frac{y}{\Delta(u-z)}\right)^2 \\ &\quad -\frac{Ay}{6\Delta}x^3+\frac{Au}{24}x^4+g(x-\frac{L}{2})+\zeta(x-\frac{L}{2}).\end{aligned}$$

Similarly, we have the following proposition for localization.

Proposition 4 *For $\delta' > 3\varepsilon$, let*

$$\begin{aligned}\mathcal{G}_u &= \{|w|>u^{3\varepsilon}\}\cup\{|y|>u^{1/2+4\varepsilon}\}\cup\{|z|>u^{1/2+4\varepsilon}\} \\ &\quad \cup\Big\{\sup_{x\notin[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|g(x)|-\delta' ux^2>0\Big\}\cup\Big\{\sup_{x\in[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|g(x)|>u^{-1/2+\delta'}\Big\}\end{aligned}$$

Under the conditions of Theorem 2, we have

$$P(\mathcal{G}_u;\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b)=o(1)e^{-u^2/2}.$$

Let

$$\mathcal{L}_u=\mathcal{G}_u^c.$$

We now proceed to the factor

$$F(x)-\frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt}.$$

Following exactly the same derivation as Lemma 2 in Section 3.1 and noting that $p(x)\equiv p_0$, we have that

$$F(x)-\frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt}=p_0\gamma\exp\left\{\frac{Ay^3}{3\Delta^4(u-z)^3\gamma}+o(u^{-1})+\omega(u)\right\},$$

where we redefine a change of variable

$$\gamma=x-\frac{L}{2}-\frac{y}{\Delta(u-z)}.$$

Thus, similar to (19), we obtain that

$$\begin{aligned}v'(x) &= e^{\sigma\xi(x)}\left[F(t)-\frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt}\right] \\ &= e^{\sigma u+\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}}\times p_0\gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2} \\ &\quad \times\exp\Big\{\frac{\sigma\Delta z}{2}\gamma^2-\frac{\sigma A}{6\Delta}y(\gamma+\frac{y}{\Delta(u-z)})^3+\frac{\sigma Au}{24}(\gamma+\frac{y}{\Delta(u-z)})^4 \\ &\quad +\frac{Ay^3}{3\Delta^4(u-z)^3\gamma}+o(u^{-1})+\omega(u)\Big\}.\end{aligned}$$

We further simplify the above display and obain that

$$\begin{aligned} v'(x) &= e^{\sigma u+\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}} \times p_0\gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2} \\ &\quad \times \exp\Big\{\frac{\sigma\Delta z}{2}\gamma^2 - \frac{\sigma A\gamma^2}{4\Delta^2 u}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + y^3\Big[\frac{\sigma A\gamma}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma}\Big] \\ &\qquad + O(u^{-1}) + \omega(u)\Big\}. \end{aligned}$$

For all $|y| \le (1+\varepsilon')\Delta u^{1/2+\varepsilon}$, we have that

$$\begin{aligned} \max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]} v'(x) &\le \max_{x\in[\frac{L}{2}-(1+2\varepsilon')u^{-1/2+\varepsilon},\frac{L}{2}+(1+2\varepsilon')u^{-1/2+\varepsilon}]} v'(x) \\ &= e^{\sigma u+\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}} \times p_0\gamma_* e^{-\frac{\sigma\Delta u}{2}\gamma_*^2} \\ &\quad \times \exp\Big\{\frac{\sigma\Delta z}{2}\gamma_*^2 - \frac{\sigma A\gamma_*^2}{4\Delta^2 u}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + y^3\Big[\frac{\sigma A\gamma_*}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma_*}\Big] \\ &\qquad + O(u^{-1}+z^2u^{-2}) + \omega(u)\Big\}. \end{aligned} \tag{34}$$

That is, $v'(x)$ is maximized when $x = \frac{L}{2} + \gamma_* + \frac{y}{\Delta(u-z)} + o(u^{-1}) + O(z\gamma_*/u)$. Since $\gamma_* = \Delta^{-1/2}\sigma^{-1/2}u^{-1/2}$, then

$$\frac{\sigma A\gamma_*}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma_*} = 0.$$

Thus, we have that

$$\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]} v'(x) \quad > \quad b$$

implies that

$$\begin{aligned} \mathcal{A} &\triangleq \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{z}{2u} - \frac{A}{4\Delta^3 u^2}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + O(z^2u^{-2}) + O(u^{-1}) \\ &\ge \omega(u). \end{aligned}$$

Corresponding to the analysis in Section 3.1, the next step is to insert $\mathcal{A}$ to $S(w,y,z)$ and obtain that

$$
\begin{aligned}
S(w,y,z) &= u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta^2}+2u(w+\frac{y^2}{2\Delta u})\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{\Delta^2}{A}z^2\\
&\quad +2u\frac{\mathcal{A}}{\sigma}-\frac{y^2z}{\Delta u}-\frac{z}{\sigma}+\frac{A}{2\Delta^3\sigma}\frac{y^2}{u}+\frac{A}{4\Delta^4}\frac{y^4}{u^2}+O(z^2/u)+O(1)\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{2u\mathcal{A}}{\sigma}\\
&\quad +\frac{\Delta^2}{A}z^2-z\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)+\frac{A}{4\Delta^2}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)^2-\frac{A}{4\Delta^2\sigma^2}+O(z^2/u)+O(1)\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{2u\mathcal{A}}{\sigma}\\
&\quad +\Big[\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)\Big]^2-\frac{A}{4\Delta^2\sigma^2}+O(u^{8\varepsilon}).
\end{aligned}
$$

For the last step in the above derivation, we use the fact that, on the set $\mathcal{L}_u$, $O(z^2/u)=O(u^{8\varepsilon})$. Thus,

$$
\begin{aligned}
&P\left(\max_{x\in[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|v'(x)|>b\right)\\
=\;& \Delta\int_{\mathcal{L}_u}h(w,y,z)P(\max_{x\in[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|v'(x)|>b|w,y,z)dwdydz\\
=\;& O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})+\frac{A}{8\Delta^2\sigma^2}}\int_{\mathcal{L}_u}P(\mathcal{A}>\omega(u))\\
&\times\exp\Big\{-\frac{u\mathcal{A}}{\sigma}-\frac{1}{2}\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}-\frac{1}{2}\Big[\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)\Big]^2\Big\}dwdydz.
\end{aligned}
$$

We introduce a change of variable

$$
B=\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}(\frac{y^2}{\Delta u}+\frac{1}{\sigma}).
$$

Then,

$$
\begin{aligned}
\sqrt{A}w+\Delta^2A^{-1/2}z &= \Delta B+\sqrt{A}w+\frac{\sqrt{A}}{2}(\frac{y^2}{\Delta u}+\frac{1}{\sigma})\\
&= \frac{\sqrt{A}}{2\sigma}+\Delta B+\sqrt{A}\mathcal{A}+o(1).
\end{aligned}
$$

Thus, by dominated convergence theorem and applying the change of variable from (w,z,y) to $(\mathcal{A},B,y)$, we have that

$$
P\left(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b;|y|\le(1+\varepsilon')\Delta u^{1/2+\varepsilon};\mathcal{L}_u\right)=O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}. \quad (35)
$$

For $|y|>(1+\varepsilon')\Delta u^{1/2+\varepsilon}$, note that the function $|v'(x)|$ is maximized at $x=\frac{L}{2}+\gamma_*+\frac{y}{\Delta(u-z)}$, that is outside the interval $[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]$. Therefore, $\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|$ is less than the estimate in (34) by at least a factor of $e^{-\lambda u^{2\varepsilon}}$ (by considering the dominating term $\gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2}$). Therefore,

$$\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b$$

if

$$\mathcal{A}=\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}+\frac{z}{2u}-\frac{A}{4\Delta^3u^2}y^2-\frac{\sigma A}{8\Delta^4u^3}y^4+O(z^2/u^2)+O(u^{-1})>\lambda u^{2\varepsilon}+\omega(u).$$

Thus,

$$\begin{aligned}&P\Big(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b;(1-\varepsilon')\Delta u^{1/2+\varepsilon}\le|y|\le u^{1/2+4\varepsilon};\mathcal{L}_u\Big) \quad (36)\\ =\;& O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.\end{aligned}$$

We combine the solution of (35), (36), Lemma 4 and obtain that

$$\begin{aligned}\alpha(u,\varepsilon) &= P\left(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b\right)\\ &= O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.\end{aligned}$$

Thus

$$P(E_1)=O(1)u^{1/2-\varepsilon}\alpha(u,\varepsilon)=O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.$$

As ε can be chosen arbitrarily small, we obtain (33) by redefining ε.

B Proofs of Propositions

Proof of Proposition 1. The proof needs a change of measure described as follows. For $\zeta\in R$, let

$$A_\zeta=\{x:\xi(x)>\zeta\}\cap[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$$

be the excursion set (on the interval $[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$) over level ζ and let P be the underlying nominal (original) probability measure. Define $Q_\zeta(\cdot)$ via

$$dQ_\zeta=\frac{mes(A_\zeta)}{E(mes(A_\zeta))}dP=\frac{mes(A_\zeta)}{\int_{x_*+u^{-1/2+\delta/2}}^{L-u^{-1/2+\delta}}P(\xi(x)>\zeta)dx}dP, \quad (37)$$

where $E(\cdot)$ is the expectation under P and $mes(A_\zeta)$ is the Lebesgue measure of the excursion set above level ζ. Note that under Q_ζ, almost surely $\sup_L\xi(x)>\zeta$. In order to generate sample paths according Q_ζ, one first simulates τ with density function $\{h(\tau):\tau\in[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]\}$

$$h(\tau)=\frac{P(\xi(\tau)>b)}{E(mes(A_\zeta))} \quad (38)$$

that is a uniform distribution over the interval $[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$; then simulate $\xi(\tau)$ conditional distribution (under the original law) given that $\xi(\tau)>\zeta$; lastly simulate $\{\xi(x):x\neq\tau\}$

given $(\tau, \xi(\tau))$ according to the original distribution. If ζ is suitably chosen, Q_ζ serves as a good approximation of the conditional distribution of $\xi(x)$ given that $\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > b$.

Lemma 10 *Under conditions in Theorem 1, we have that*

$$P\Big(\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2, E_1\Big) = o(u^{-1}e^{-u^2/2}).$$

Proof of Lemma 10. Let

$$F_b = \{\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2\}.$$

Let $\zeta = u - (\log u)^2 - 1/u$. Then, the probability can be written as

$$\begin{aligned} P\left(F_b, E_1\right) &\leq O(1)E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; F_b, E_1\right] \\ &= O(1)\int_{x_*+u^{-1/2+\delta/2}}^{L-u^{-1/2+\delta}} E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; F_b, E_b\right] d\tau, \end{aligned}$$

where we use E_τ^Q to denote the conditional expectation $E^Q(\cdot|\tau)$ under the measure Q_ζ. Given a particular $\tau \in [x_* + u^{-1/2+\delta/2}, L - u^{-1/2+\delta}]$, we redefine the change of variables

$$\xi(\tau) = u + w, \xi'(\tau) = y, \xi''(\tau) = -\Delta(u - z).$$

Note that the current definition of (w, y, z) is different from that in the proposition and Theorem 1. As the previous definition of (w, y, z) will not be used in this lemma, to simplify the notation, we do not create another notation and use (w, y, z) differently. Conditional on (w, y, z) the process $g(x)$ is a mean zero Gaussian process such that

$$\xi(x) = E(\xi(x)|w, y, z) + g(x - \tau).$$

We have the bound of the excursion set that $E^Q(1/mes(A_\zeta)) = O(u)$, the detailed development of which is omitted. With this in mind, we first have that that

$$E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |z| \geq u^{1/2+\delta/16}, F_b, E_b\right] = o(u^{-1}e^{-u^2/2}).$$

and similarly

$$E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y| \geq u^{1/2+\delta/16}, F_b, E_b\right] = o(u^{-1}e^{-u^2/2}).$$

In addition, for some λ_0 sufficiently large and δ_0 small, we have that

$$\begin{aligned} &E\left(\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; \sup_{|x|\leq u^{-1/2+\delta}} |g(x)| > \lambda_0 u^{-1+4\delta}, \text{ or } \sup_{|x|>u^{-1/2+\delta}} |g(x)| - \delta_0 u x^2 > 0\right) \\ = &\ o(u^{-1}e^{-u^2/2}). \end{aligned}$$

Then, we only need to consider the situation that $|y| < u^{1/2+\delta/16}$ and $|z| < u^{1/2+\delta/16}$. Furthermore, using Taylor expansion on $\xi(x)$ as we had done several times previously, the process $\xi(x)$ is a approximately a quadratic function with mode being $\tau + \frac{y}{\Delta\sigma(u-z)}$ for $\tau \in [x_* + u^{-1/2+\delta/2}, L - u^{-1/2+\delta}]$. Thus, when considering the integral $\int_0^L e^{\xi(t)}dt$ and $\int_0^L (F(x) - F(t))e^{\xi(t)}dt$, we do not have to consider the boundary issue as in the analysis of $P(E_2)$. With the same calculations for (21) by expanding ξ at τ instead of x_*, we obtain that

$$\sup_{x\in[u^{-1/2+\delta}, L-u^{-1/2+\delta}]} |v'(x)| \geq b$$

if and only if

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad - \frac{\sigma A}{6\Delta} y(\gamma_* + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24}(\gamma_* + \frac{y}{\Delta(u-z)})^4 \\
&\quad - \frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) \\
&\quad - \frac{Ay^3}{\Delta^4(u-z)^3\gamma_*} + \log\frac{p(x)}{p(x_*)} \\
&\geq o(u^{-1}) + \omega(u),
\end{aligned}$$

where the x in "$p(x)$" is $x = \tau + \gamma_* + \frac{y}{\Delta(u-z)} + o(u^{-1}) + O(z\gamma_*/u)$. Similar to the derivation for (39), we expand the second row in the definition of $\mathcal{A}$ and obtain that

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z - \frac{\sigma A y^4}{8\Delta^4(u-z)^3} + \frac{\sigma\Delta z}{2}\gamma_*^2 - \frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2 + \frac{\sigma A(u-z)}{24}\gamma_*^4 \\
&\quad - \frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x_*)}{6p(x_*)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) + \log\frac{p(x)}{p(x_*)}.
\end{aligned}$$

Notice that

$$\frac{p''(x)}{6p(x)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) = O(u^{-1}).$$

When $|x - x_*| < \varepsilon$, by Taylor expansion

$$|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})| = O((x-x_*)/\sqrt{u}) = o(\log p(x) - \log p(x_*));$$

when $|x - x_*| > \varepsilon$

$$|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})| = O(u^{-1/2}) = o(1) = o(\log p(x) - \log p(x_*)).$$

Therefore $|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})|$ is always of a smaller order than $\log p(x) - \log p(x_*)$. On the region $|x - x_*| > \frac{u^{-1/2+\delta/2}}{2}$, there exists a positive λ such that

$$\log\frac{p(x)}{p(x_*)} \leq -2\lambda u^{-1+\delta}.$$

Thus, $\mathcal{A}$ is bounded by

$$\begin{aligned}\mathcal{A} < \mathcal{A}' &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2} y^2 z - \frac{\sigma A y^4}{8\Delta^4 (u-z)^3} + \frac{\sigma \Delta z}{2}\gamma_*^2 - \frac{\sigma A y^2}{4\Delta^2 (u-z)}\gamma_*^2 + \frac{\sigma A(u-z)}{24}\gamma_*^4 \\ &\quad -\lambda u^{-1+\delta}\end{aligned}$$

Furthermore, notice that

$$\begin{aligned}& E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y|, |z| \le u^{1/2+\delta/16}, F_b, E_1\right] \\ \le\ & O(1)\int_{w\ge -(\log u)^2} e^{-\frac{1}{2}S(w,y,z)} \frac{P(\mathcal{A}' \ge \omega(u), F_b)}{mes(A_\zeta)} dwdydz.\end{aligned}$$

Similar to the previous development, we write

$$\begin{aligned}S(w,y,z) &= u^2 + w^2 + \frac{\Delta^2 (w-z)^2}{A - \Delta^2} + 2u(w + \frac{y^2}{2\Delta u}) \\ &= u^2 + w^2 + \frac{\Delta^2 (w-z)^2}{A - \Delta^2} \\ &\quad + 2u\Big[\frac{\mathcal{A}'}{\sigma} - \frac{y^2 z}{2\Delta u^2} + \frac{A y^4}{8\Delta^4 (u-z)^3} - \frac{\Delta z}{2}\gamma_*^2 + \frac{A y^2}{4\Delta^2 (u-z)}\gamma_*^2 - \frac{A(u-z)}{24}\gamma_*^4 + \lambda u^{-1+\delta}/\sigma\Big].\end{aligned}$$

Thus, by dominated convergence theorem and the fact that $mes(A_\zeta)^{-1} = O(u)$, we have that

$$\begin{aligned}& E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y|, |z| \le u^{1/2+\delta/16}, F_b, E_1\right] \\ \le\ & O(1)\int_{|y|,|z|\le u^{-1/2+\varepsilon/4}} E(mes(A_\zeta)^{-1}; \mathcal{A}' \ge \omega(u)) e^{-\frac{1}{2}S(w,y,z)} dwdydz \\ \le\ & O(1) e^{-\frac{u^2}{2} - \lambda u^\delta/\sigma} \\ & \times \int_{|y|,|z|\le u^{-1/2+\varepsilon/4}} E(mes(A_\zeta)^{-1}; \mathcal{A}' \ge \omega(u)) \\ & \qquad \times \exp\left[-\frac{\Delta^2}{2(A-\Delta^2)} z^2 - \frac{u\mathcal{A}'}{\sigma} + \frac{y^2 z}{2\Delta u} - \frac{A y^4}{8\Delta^4 u^2} + \frac{z}{2\sigma} + \frac{A y^2}{4\Delta^3 \sigma u}\right] dwdydz \\ =\ & o(u^{-1} e^{-u^2/2}).\end{aligned}$$

■

With a completely analogous proof as the Lemma 10, we have that

Lemma 11 *Under conditions in Theorem 1, we have that*

$$P\left(\sup_{x\in[u^{-1/2+\delta}, x_* - u^{-1/2+\delta/2}]} \xi(x) > u - (\log u)^2, E_1\right) = o(u^{-1} e^{-u^2/2}).$$

We write

$$J_b = \{\sup_{x\in[u^{-1/2+\delta}, x_* - u^{-1/2+\delta/2}]} \xi(x) > u - (\log u)^2\} \cup \{\sup_{x\in[\tau_* + u^{-1/2+\delta/2}, L - u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2\}$$

and thus

$$P(J_b^c, E_1) = o(u^{-1}e^{-u^2/2}).$$

We proceed to the following lemma to complete the proof of the proposition.

Lemma 12 *Let (w, y, z) defined as in Section 3.1. For $\varepsilon > 0$, let*

$$L_b = \{|w| < u^{3\delta}, |y| < u^{1/2+4\delta}, |z| < u^{1/2+4\delta}\}$$

Under conditions of Theorem 1, we have that

$$P\left(L_b^c, J_b^c, E_1\right) = o(u^{-1}e^{-u^2/2}).$$

Proof. Note that $|v'(x)| > b$ implies that $\xi(x) > \log b - \kappa_0 = u - O(\log u)$ for some $\kappa_0 > 0$. Thus, on the set J_b^c, E_1 implies that $\sup_{[x_* - u^{-1/2+\delta/2}, x_* + u^{-1/2+\delta/2}]} \xi(x) > \frac{\log b}{\sigma} - (\log u)^2$. Therefore, we have that

$$P(|w| > u^{3\delta}, F_b^c, E_b) \le P(|w| > u^{3\delta}, \sup_{[x_* - u^{-1/2+\delta/2}, x_* + u^{-1/2+\delta/2}]} \xi(x) > \frac{\log b}{\sigma} - (\log u)^2) = o(u^{-1}e^{-u^2/2}),$$

where the last step is an application of Borel-TIS lemma ([6, 15, 4]). Furthermore, by simply bound of Gaussian distribution, we have that

$$P(|w| < u^{3\delta}, |z| > u^{1/2+4\delta}, F_b^c, E_b) = o(u^{-1}e^{-u^2/2}),$$

and

$$P(|w| < u^{3\delta}, |y| > u^{1/2+4\delta}, F_b^c, E_b) = o(u^{-1}e^{-u^2/2}).$$

We thus conclude the proof. ■

The results of Lemmas 10, 11, and 12 immediately lead to the conclusion of Proposition 1. ■

Proof of Proposition 2. Note that $g(x)$ is independent of (w, y, z) and $\mathcal{L}_u$ only depends on (w, y, z). Therefore,

$$\begin{aligned} P\left(\sup_{|x| > u^{-1/2+8\delta}} [|g(x)| - \delta' u x^2] > 0,\ \mathcal{L}_u\right) &= P\left(\sup_{|x| > u^{-1/2+8\delta}} [|g(x)| - \delta' u x^2] > 0\right) P(\mathcal{L}_u) \\ &= o(u^{-1}e^{-u^2/2}). \end{aligned}$$

The last step is a direct application of the Borel-TIS lemma and the fact that $P(\mathcal{L}_u) = O(e^{-u^2/2 + O(u^{1+3\delta})})$. With a similar argument, we obtain the second bound. ■

Proof of Propositions 3 and 4. The proofs of these two propositions are completely analogous to that of Proposition 1, that is, basically a repeated application of Borel-TIS lemma and the change of measure Q_ζ. Therefore, we omit the details. ■

C Proof of the Lemmas

Proof of Lemma 1. On the set $|x - x_*| < u^{-1/2+8\delta}$ and $\mathcal{L}_u'$, we have $s = O(u^{8\delta})$ and thus

$$\frac{y^3 s}{(u-z)^{5/2}} = O(u^{-1+20\delta}), \frac{y^2 s^2}{(u-z)^2} = O(u^{-1+24\delta}), \frac{s^4}{(u-z)} = O(u^{-1+32\delta}).$$

Let X be a standard Gaussian random variable. We conclude the proof by the following calcuation

$$
\begin{aligned}
& \int_{|x-x_*|<u^{-1/2+8\delta}} e^{\sigma[-\frac{s^2}{2}-\frac{Ay^3}{\Delta^{7/2}(u-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{A}{24\Delta^2(u-z)}s^4]}ds \\
= \;& e^{o(u^{-1})}\int_{|x-x_*|<u^{-1/2+8\delta}} e^{-\frac{\sigma s^2}{2}}\times\left(1-\frac{\sigma Ay^3}{\Delta^{7/2}(u-z)^{5/2}}s-\frac{\sigma Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{\sigma A}{24\Delta^2(u-z)}s^4\right)ds \\
= \;& e^{o(u^{-1})}\sqrt{\frac{2\pi}{\sigma}}E\left[1-\frac{A\sigma^{1/2}y^3X}{\Delta^{7/2}(u-z)^{5/2}}-\frac{Ay^2X^2}{4\Delta^3(u-z)^2}+\frac{AX^4}{24\Delta^2\sigma(u-z)}\right] \\
= \;& \sqrt{\frac{2\pi}{\sigma}}\exp\left\{-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma(u-z)}+o(u^{-1})\right\} \\
= \;& \sqrt{\frac{2\pi}{\sigma}}\exp\left\{-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma u}+o(u^{-1})\right\}.
\end{aligned}
$$

■

Proof of Lemma 2. We use the result of Lemma 1 and the Taylor expansion

$$F(x)-F(t)=p(x)(x-t)-\frac{1}{2}p'(x)(x-t)^2+\frac{1}{6}p''(x)(x-t)^3+o(x-t)^4.$$

Recall the change of variable

$$s(t)=\sqrt{\Delta(u-z)}\left(t-x_*-\frac{y}{\Delta(u-z)}\right)$$

at the beginning of Step 1 of the main proof. We apply it to the spatial index t. Note that $t-x_*-s(t)/\sqrt{\Delta(u-z)}=y/(\Delta(u-z))$ and $x-t=\gamma-s(t)/\sqrt{\Delta(u-z)}$. We perform the same splitting as in (15), insert the result in (16), use the expansion of ξ in (13), and obtain that

$$
\begin{aligned}
& \left(\int_0^L e^{\sigma\xi(t)}dt\right)^{-1}\int_0^L (F(x)-F(t))e^{\sigma\xi(t)}dt \\
= \;& \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2}-\frac{A}{8\Delta^2\sigma(u-z)}+\omega(u)+o(u^{-1})\right\} \\
& \times\int_{|s|\le u^{8\delta}}\left[p(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)-\frac{1}{2}p'(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)^2\right. \\
& \qquad\left.+\frac{1}{6}p''(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)^3+o(u^{-3/2})\right] \\
& \times\sqrt{\frac{\sigma}{2\pi}}e^{\sigma[-\frac{s^2}{2}-\frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{A}{24\Delta^2(u-z)}s^4]}ds
\end{aligned}
$$

We rewrite the above integral by pulling out the Gaussian density and expanding the exponential term in the last row

$$
\begin{aligned}
= \quad & \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
& \times \int_{|s|\le u^{8\delta}|} \sqrt{\frac{\sigma}{2\pi}} e^{-\frac{\sigma s^2}{2}} \\
& \times \left[p(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big) - \frac{1}{2}p'(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big)^2 + \frac{1}{6}p''(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big)^3\right] \\
& \times \left[1 - \frac{\sigma A y^3}{3\Delta^{7/2}(u-z)^{5/2}}s - \frac{\sigma A y^2}{4\Delta^3(u-z)^2}s^2 + \frac{\sigma A}{24\Delta^2(u-z)}s^4\right] ds.
\end{aligned}
$$

Similar to Lemma 1, we further evaluate the above integral by computing moments of $N(0, \sigma^{-1/2})$ and obtain that (we omit several cross terms that can be absorbed by $o(u^{-1})$)

$$
\begin{aligned}
& F(x) - \frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt} \\
= \quad & \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
& \times\left[p(x)\gamma - \frac{p'(x)}{2}\left(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}\right) + \frac{p''(x)}{6}\left(\gamma^3 + \frac{3\gamma}{\sigma\Delta(u-z)}\right)\right. \\
& \qquad \left. + p(x)\frac{Ay^3}{3\Delta^4(u-z)^3} - p(x)\gamma\frac{Ay^2}{4\Delta^3(u-z)^2} + p(x)\gamma\frac{A}{8\sigma\Delta^2(u-z)}\right].
\end{aligned}
$$

We take out the factor "$p(x)\gamma$" from the bracket and continue the calculation

$$
\begin{aligned}
= \quad & \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
& \times p(x)\gamma \exp\Big[- \frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\
& \qquad + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} - \frac{Ay^2}{4\Delta^3(u-z)^2} + \frac{A}{8\sigma\Delta^2(u-z)}\Big].
\end{aligned}
$$

We further simplify the above display and obtain that

$$
\begin{aligned}
= \quad & p(x)\gamma \exp\Big[- \frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\
& \qquad + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} + o(u^{-1}) + \omega(u)\Big].
\end{aligned}
$$

■

Proof of Lemma 3. Let $\mathcal{A}$ be defined as in (21). Note that $p'(x_*) = 0$ and $p'(x) \sim p''(x_*)(\gamma + y/\Delta(u-z))$. We apply Taylor expansion of the term $\log \frac{p(x_* + \gamma_* + \Delta^{-1}(u-z)^{-1}y)}{p(x_*)}$ in (21) and expand the second row of (21). Thus, $\mathcal{A}$ can be further simplified to

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z - \frac{\sigma A y^4}{8\Delta^4(u-z)^3} + \frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad - \frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_* - \frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2 + \frac{\sigma A u}{24}\gamma_*^4 \\
&\quad - \frac{p''(x_*)}{2p(x_*)}(\gamma_* + \frac{y}{\Delta(u-z)})(\gamma_* + \frac{1}{\sigma\Delta(u-z)\gamma_*}) \\
&\quad + \frac{p''(x_*)}{6p(x_*)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma_*} \\
&\quad + \frac{p''(x_*)}{2p(x_*)}(\gamma_* + \frac{y}{\Delta(u-z)})^2 + o(y^2u^{-2}) + O(z^2/u^2).
\end{aligned}$$

Note that $\gamma_* = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. The term

$$-\frac{p''(x_*)}{2p(x_*)}\frac{y}{\Delta(u-z)}(\gamma_* + \frac{1}{\sigma\Delta(u-z)\gamma_*})$$

expanded from the third row cancels the cross term

$$\frac{\gamma_* p''(x_*)}{p(x_*)}\frac{y}{\Delta(u-z)}$$

expanded from the quadratic term in the last row. Then, $\mathcal{A}$ is further simplified to

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z - \frac{\sigma A y^4}{8\Delta^4(u-z)^3} + \frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad - \frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_* - \frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2 + \frac{\sigma A(u-z)}{24}\gamma_*^4 \\
&\quad - \frac{p''(x_*)}{2p(x_*)}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) \\
&\quad + \frac{p''(x_*)}{6p(x_*)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma_*} \\
&\quad + \frac{p''(x_*)}{2p(x_*)}(\gamma_*^2 + \frac{y^2}{\Delta^2(u-z)^2}) + o(y^2u^{-2}) + O(z^2/u^2).
\end{aligned} \tag{39}$$

Furthermore, the term $-\frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_*$ in the second row cancels $\frac{Ay^3}{3\Delta^4(u-z)^3\gamma_*}$ in the fourth row. We now plug in $\gamma_*^2 = \Delta^{-1}\sigma^{-1}u^{-1}$ and obtain that

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z - \frac{\sigma A y^4}{8\Delta^4 u^3} + \frac{z}{2u} \\
&\quad - \frac{Ay^2}{4\Delta^3 u^2} + \frac{A}{24\sigma\Delta^2 u} - \frac{p''(x_*)}{3p(x_*)\sigma\Delta u} + \frac{p''(x_*)}{2p(x_*)}(\frac{1}{\sigma\Delta u} + \frac{y^2}{\Delta^2 u^2}) + o(u^{-1}) + O(z^2/u) \\
&= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z + \frac{z}{2u} + \frac{A}{24\sigma\Delta^2 u} + \frac{p''(x_*)}{6p(x_*)\sigma\Delta u} \\
&\quad - \frac{\sigma A y^4}{8\Delta u^3} + \frac{y^2}{u^2}(-\frac{A}{4\Delta^3} + \frac{p''(x_*)}{2p(x_*)\Delta^2}) + o(u^{-1} + y^2u^{-2}) + O(z^2/u^2).
\end{aligned}$$

■

Proof of Lemma 5. By simple algebra, we have that

$$\begin{aligned} S(w,y,z) &= u^2 + 2u\mathcal{A}/\sigma + \frac{(\sqrt{A}w + \Delta^2 A^{-1/2}z)^2}{A-\Delta^2} + \frac{\Delta^2}{A}z^2 \\ &\quad -\frac{y^2 z}{\Delta u} - \frac{z}{\sigma} + \frac{A}{4\Delta^4 u^2}y^4 - \frac{y^2}{u}\Big(-\frac{A}{2\sigma\Delta^3} + \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\Big) + o(y^2/u) + O(z^2/u) + O(1) \\ &= u^2 + 2u\mathcal{A}/\sigma + \frac{(\sqrt{A}w + \Delta^2 A^{-1/2}z)^2}{A-\Delta^2} + \frac{\Delta^2}{A}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big)^2 \\ &\quad + \frac{1}{\sigma}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big) - \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\frac{y^2}{u} + o(y^2/u) + O(z^2/u) + O(1). \end{aligned}$$

Note that, on the set $\mathcal{L}'_u$, $o(y^2/u) + O(z^2/u) = o(y^2/u + z)$ and thus,

$$S(w,y,z) \geq u^2 + 2u\mathcal{A}/\sigma + \frac{\Delta^2}{A}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big)^2 + \frac{1+o(1)}{\sigma}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big) - \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\frac{y^2}{u} + O(1).$$

■

Proof of Lemma 6. Using the second change of variable in (24), the denominator in (25) is

$$\int_0^L e^{\sigma\xi(t)}dt = e^{c_*}\int_0^L \exp\Big\{\sigma\Big[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}u_L^{5/2}}s - \frac{Ay^2}{4\Delta^3 u_L^2}s^2 + \frac{A}{24\Delta^2 u_L}s^4\Big]\Big\}dt.$$

Let Z be a standard Gaussian random variable following $N(0,1)$. With a similar splitting in (15) and the derivation in Lemma 1 and noticing the boundary constraint that

$$t \leq L \Longleftrightarrow s \leq \sqrt{\frac{(1-z/u_L)}{\sigma}}\zeta_L - \frac{y}{\sqrt{\Delta(u_L - z)}},$$

we apply Taylor expansion on the integrand and have that

$$\begin{aligned} &= \frac{\sqrt{2\pi}e^{c_* + o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L - z)}}e^{\omega(u_L)} \\ &\quad \times E\left[1 - \frac{\sigma^{1/2}Ay^3}{3\Delta^{7/2}u_L^{5/2}}Z - \frac{Ay^2}{4\Delta^3 u_L^2}Z^2 + \frac{A}{24\Delta^2\sigma u_L}Z^4; Z \leq \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\right] \\ &= \frac{\sqrt{2\pi}e^{c_* + o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L - z)}}e^{\omega(u_L) + O(y^3/u_L^{5/2} + y^2/u_L^2)} \\ &\quad \times E\left[1 + \frac{A}{24\Delta^2\sigma u_L}Z^4; Z \leq \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\right], \end{aligned}$$

where $c_* = \sigma(u_L + w + \frac{y^2}{2\Delta(u_L - z)} - \frac{Ay^4}{8\Delta^4(u_L - z)^3})$ and $\omega(u) = O(\sup_{|x|\leq u^{-1/2+8\delta}}|g(x)|)$. The expectation in the previous display can be written as

$$\begin{aligned} & E\left[1+\frac{A}{24\Delta^2\sigma u_L}Z^4; Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\right] \\ = \ & P\Big[Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big] \\ & \times\exp\left\{\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)+\omega(u_L)+O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})\right\}\end{aligned}$$

We use the fact that $E(Z^4|Z\le\zeta_L)=E(Z^4|Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y)+o(1+yu^{-1/2})$. We continue the calculations and obtain that

$$\begin{aligned}\int_0^L e^{\sigma\xi(t)}dt \ = \ & \frac{\sqrt{2\pi}e^{c_*+o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L-z)}}P\Big[Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big] \\ & \times\exp\left\{\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)+\omega(u_L)+O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})\right\}.\end{aligned}$$

We now proceed to the numerator of (25). Using Taylor expansion

$$F(x)-F(t)=p(x)(x-t)-\frac{1}{2}p'(x)(x-t)^2+\frac{1}{6}p''(x)(x-t)^3+o(x-t)^3,$$

the numerator of (25) is (with the splitting as in (15))

$$\begin{aligned} & \int_0^L(F(x)-F(t))e^{\sigma\xi(t)}dt \\ = \ & \frac{e^{c_*+\omega(u_L)+o(u_L^{-1})}}{\sqrt{\Delta(u_L-z)}}\times\int_{-u^{8\delta}}^{\sqrt{\frac{(1-z/u)}{\sigma}}\zeta_L-\frac{y}{\sqrt{\Delta(u-z)}}} \\ & \left[p(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})-\frac{1}{2}p'(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})^2+\frac{1}{6}p''(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})^3+o(u_L^{-3/2})\right] \\ & \times e^{\sigma\left\{-\frac{s^2}{2}-\frac{Ay^3}{3\Delta^{7/2}(u_L-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u_L-z)^2}s^2+\frac{A}{24\Delta^2(u_L-z)}s^4\right\}}ds \\ = \ & \sqrt{\frac{2\pi}{\Delta\sigma(u_L-z)}}e^{c_*+\omega(u_L)+o(u_L^{-1})} \\ & \times E\Big\{p(x)(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})-\frac{p'(x)}{2}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^2+\frac{p''(x)}{6}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^3 \\ & \quad+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}}) \\ & \quad+O(y^3/u_L^{5/2}+y^2/u_L^2+u_L^{-2}) \ ; \ \ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}.\end{aligned}$$

Thus, the factor in (25) is

$$
\begin{aligned}
&\int_0^L (F(x)-F(t))\frac{e^{\sigma\xi(t)}}{\int_0^L e^{\sigma\xi(s)}ds}dt \\
= \quad & \exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\} \\
&\times E\Big\{p(x)(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})-\frac{p'(x)}{2}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^2+\frac{p''(x)}{6}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^3 \\
&\quad +\frac{Ap(x)}{24\Delta^2\sigma^2 u_L}Z^4(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}
\end{aligned}
$$

where $\lambda(u_L)=O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})+o(u_L^{-1}+u_L^{-1}z)$. We take out a factor $\sqrt{\Delta\sigma(u_L-z)}$ from the above expectation and obtain that

$$
\begin{aligned}
= \quad & \exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\} \\
&\frac{1}{\sqrt{\Delta\sigma u_L(1-z/u_L)}}E\Big\{p(x)(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)-\frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)^2 \\
&+\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}.
\end{aligned}
$$

Notice that in the last two terms of the above display and for the denominator of the second term in the second low, "u_L-z" is replaced by u_L. The error caused by this change can be absorbed into $\lambda(u_L)$. Notice that

$$
\frac{1}{\sqrt{\Delta\sigma u_L(1-z/u_L)}}=\frac{e^{\frac{z}{2u_L}+o(z/u_L)}}{\sqrt{\Delta\sigma u_L}}.
$$

We further separate the expectation into two parts and obtain that

$$
\begin{aligned}
= \quad & \exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\}\times\frac{e^{\frac{z}{2u_L}}}{\sqrt{\Delta\sigma u_L}} \\
&\times\Big\{E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z) \\
&\qquad -\frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)^2\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big] \\
&\quad +E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3 \\
&\qquad +\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\Big\}.
\end{aligned}
$$

Thus, we conclude the proof. ■

Proof of Lemma 7. Similar to the calculations resulting (18), we obtain that

$$\begin{aligned}
\xi(x) &= u_L + w + \frac{y^2}{2\Delta(u_L - z)} - \frac{\Delta(u_L - z)}{2}\gamma^2 - \frac{A}{6\Delta}y(\gamma + \frac{y}{\Delta(u_L - z)})^3 + \frac{Au_L}{24}(\gamma + \frac{y}{\Delta(u_L - z)})^4 \\
&\quad + g(x - t_L) + \vartheta(x - t_L) \\
&= u_L + w + \frac{y^2}{2\Delta u_L} - \frac{\Delta(u_L - z)}{2}\gamma^2 + \frac{Au_L}{24}\gamma^4 + o(u^{-1}y^2) + g(x - t_L) + \vartheta(x - t_L),
\end{aligned}$$

where $\vartheta(x) = O(u^{1/2+4\delta}x^5 + ux^6)$. Combining the above expression and Lemma 6, we obtain that

$$\begin{aligned}
v'(x) &= e^{\sigma\xi(x)}\int_0^L (F(x) - F(t))\frac{e^{\sigma\xi(t)}}{\int e^{\sigma\xi(s)}ds}dt \\
&= \exp\Big\{\lambda(u_L) + O(y^2 z u_L^{-2}) + \omega(u_L) + \sigma u_L + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{A\sigma u_L}{24}\gamma^4\Big\} \\
&\quad \times \frac{1}{\sqrt{\Delta\sigma u_L}}\exp\Big\{ - \frac{\sigma\Delta(u_L - z)}{2}\gamma^2 + \frac{z}{2u_L} - \frac{A}{24\Delta^2\sigma u_L}E\left(Z^4 | Z \le \zeta_L\right)\Big\} \\
&\quad \Big\{E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big|\ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\Big] \\
&\quad + E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 \\
&\qquad + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z)\ \Big|\ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\Big]\Big\}
\end{aligned}$$

Using Taylor expansion on the two expectation terms, we obtain that

$$\begin{aligned}
&E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big|\ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\Big] \\
&\quad + E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 \\
&\qquad + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z)\ \Big|\ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\Big] \\
= \ &E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big|\ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\Big] \\
&\times \exp\left\{\frac{E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z)\,|Z \le \zeta_L\Big]}{p(x)E(\gamma\sqrt{\sigma\Delta u_L} - Z|Z \le \zeta_L)} + o(u_L^{-1} + yu_L^{-1})\right\}.
\end{aligned}$$

We insert the above identity back to the expression of $v'(x)$ and obtain that

$$\begin{aligned} v'(x) &= \exp\Big\{\lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\omega(u_L)+\sigma u_L+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\sigma u_L}{24}\gamma^4\Big\}\\ &\quad\times\frac{1}{\sqrt{\Delta\sigma u_L}}\exp\Big\{\frac{z}{2u_L}-\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)\Big\}\\ &\quad\times H_{L,x}\Big(\gamma\sqrt{\sigma\Delta(u_L-z)},\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y;u_L\Big)\\ &\quad\times\exp\left\{\frac{E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\,|Z\le\zeta_L\Big]}{p(x)E(\gamma\sqrt{\sigma\Delta u_L}-Z|Z\le\zeta_L)}\right\}, \end{aligned}$$

where

$$H_{L,y}(x,\zeta;u)\triangleq e^{-\frac{x^2}{2}}\times E\Big[p(y)(x-Z)-\frac{p'(y)}{2\sqrt{\Delta\sigma u}}(x-Z)^2\;\Big|\;Z\le\zeta\Big].$$

■

Proof of Lemma 8. We insert $\gamma_L=\frac{\zeta_L}{\sqrt{\sigma\Delta u_L}}$ to the expression of $\mathcal{A}$ in (31) and obtain that

$$\begin{aligned} \mathcal{A} &= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\zeta_L^4}{24\Delta^2\sigma u_L}+\frac{z}{2u_L}-\frac{AE\left(Z^4|Z\le\zeta_L\right)}{24\Delta^2\sigma u_L}\\ &\quad+G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y;u_L\Big)-G_L(\zeta_L;u_L)\\ &\quad+\frac{E[\frac{p''(L)}{6\sigma\Delta u_L}(\zeta_L-Z)^3+\frac{Ap(L)}{24\Delta^2\sigma^2u_L}Z^4(\zeta_L-Z)\,|Z\le\zeta_L]}{p(L)E(\zeta_L-Z\,|Z\le\zeta_L)}. \end{aligned}$$

Note that $\Xi_L=-\lim_{u_L\to\infty}\partial_\zeta^2G_L(\zeta_L,u_L)$. Then,

$$\begin{aligned} \mathcal{A} &= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\zeta_L^4}{24\Delta^2\sigma u_L}+\frac{z}{2u_L}-\frac{AE\left(Z^4|Z\le\zeta_L\right)}{24\Delta^2\sigma u_L}\\ &\quad+\frac{E[\frac{p''(L)}{6\sigma\Delta u_L}(\zeta_L-Z)^3+\frac{Ap(L)}{24\Delta^2\sigma^2u_L}Z^4(\zeta_L-Z)\,|Z\le\zeta_L]}{p(L)E(\zeta_L-Z|Z\le\zeta_L)}\\ &\quad-\frac{\Xi_L+o(1)}{2}\Big(\frac{\zeta_Lz}{2u_L}+\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2.\\ &= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{z}{2u_L}+\frac{\kappa_L}{u_L}-\frac{\Xi_L+o(1)}{2}\Big(\frac{\zeta_Lz}{2u_L}+\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2. \end{aligned}$$

where κ_L is given as in (9). ■

Proof of Lemma 9. In this case that $|\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y-\zeta_L|>\varepsilon$, the maximum of $|v'(x)|$ is not necessarily attained at $x=L$. Note that this does not change the calculation very much except that the terms $p(x)$ and $p'(x)$ in $H_{x,L}$ may not be evaluated on the boundary $x=L$, but still in the region $[L-u^{-1/2+\delta}, L]$. Therefore, maximizing (29), we have that

$$\begin{aligned}
& \sup_{x\in[L-u^{-1/2+\delta},L]} \log \left| H_{L,x}(\gamma\sqrt{\sigma\Delta(u_L-z)}, \sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y; u_L)\right| \\
= & \; G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y; u_L\Big) + O(u^{-1/2+\delta}).
\end{aligned}$$

Therefore, we only need to add an $O(u^{-1/2+\delta})$ to the definition of $\mathcal{A}$ in (31). Furthermore, the term in (31) is bounded by

$$G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y, u_L\Big) - G_L(\zeta_L, u_L) \le -\delta_0\varepsilon^2$$

for some $\delta_0 > 0$. Furthermore, on the set $\mathcal{L}_u^*$ we have that $\lambda(u_L) + o(yu_L^{-1}) + O(y^2 z u_L^{-2}) = o(1)$. Therefore, we have the bound $S(w,y,z) \ge u_L^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + 2u_L\mathcal{A}/\sigma + \delta_0\varepsilon^2 u_L$ and further

$$P\left(\max_{x\in[L-u_L^{-1/2+\delta},L]} |v'(x)| > b; \mathcal{L}_{u_L}^*; \quad \left|\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y - \zeta_L\right| \ge \varepsilon\right) \quad = \quad o(1)u_L^{-1}e^{-u_L^2/2}.$$

■

D Numerical Examples

In this section, we present one numerical example. We consider the differential equation in $[0, 2.5]$, that is, $L = 2.5$. The Gaussian process has zero mean and unit variance. The covariance function is

$$C(t) = e^{-t^2/2}$$

and thus $\xi(x)$ is infinitely differentiable. Furthermore, we consider a constant force $p(x) = 1$ and thus $F(x) = x$. We compute the tail probability $P(\max_{x\in[0,L]} |v'(x)| > b)$ via the approximations in the theorems, denoted by $\tilde{w}(b)$, and furthermore we compute the probabilities via importance sampling, denoted by $\hat{w}(b)$. For the Monte Carlo estimator, we choose the sample sizes such that the estimated standard deviations of the estimator is at the most 10% of $\hat{w}(b)$. Figure 2 shows the ratio between $\tilde{w}(b)/\hat{w}(b)$ as a function of $\log(b)$. The ratio stabilizes to one as b becomes large, but the convergence is quite slow as the smallest probability in Figure 2 is on the order of 10^{-9}.

We further consider a nonconstant force term $p(x) = \max(10 - 10(x-2.5)^2, 1)$ in the interval $[0, 5]$. The covariance function is $C(t) = e^{-0.3t^2}$. The corresponding plot of $\tilde{w}(b)/\hat{w}(b)$ versus $\log(b)$ is given by Figure 3. The empirical rate of convergence of the non-constant case is much slower than that of the constant case.

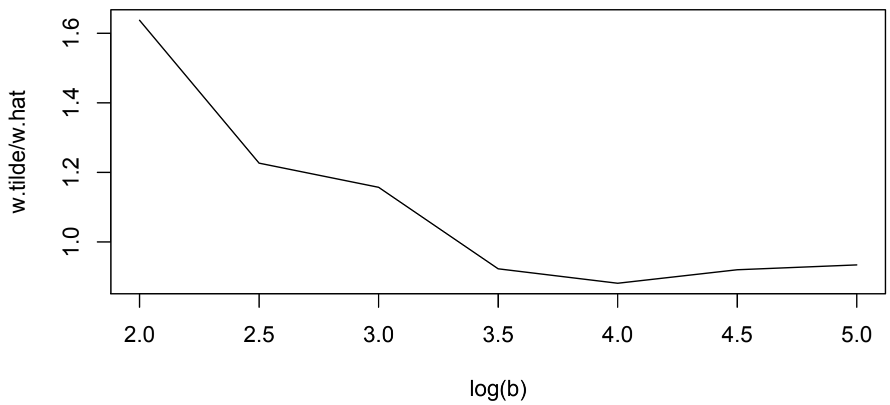

Figure 2: The ration $\tilde{w}(b)/\hat{w}(b)$ versus $\log(b)$ for $p(x) = 1$

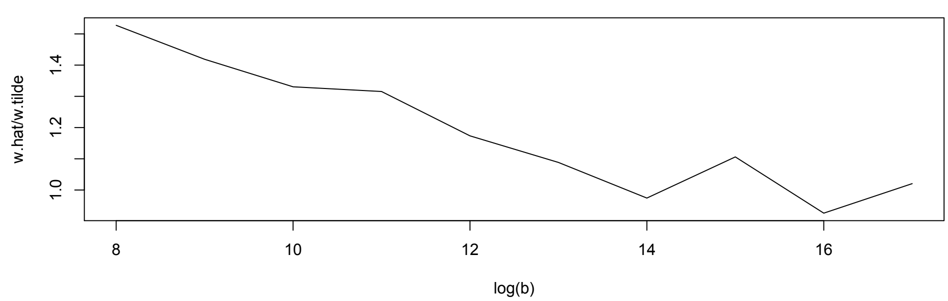

Figure 3: The ration $\hat{w}(b)/\tilde{w}(b)$ versus $\log(b)$ for non-constant $p(x)$.