Supplemental Material

A Proof of Theorem 2

Similar to the proof of Theorem 1, we consider the event E_1, E_2, and E_3 separately. By homogeneity and symmetry, $P(E_2) = P(E_3)$. The approximations of $P(E_2)$ and $P(E_3)$ are identical to those obtained in Section 3.2 by setting $p(x) \equiv p_0$. Therefore,

$$P(E_2) = P(E_3) \sim D_h u_h^{-1} e^{-u_h^{-2}/2}.$$

From the derivation of $P(E_2)$ in the previous proof, we obtain that $P(E_2 \cap E_3) = o(P(E_2))$. For the rest of the proof, we show that $P(E_1) = o(P(E_2))$ and thus $P(E_1 \cap E_2) = o(P(E_2))$.

Approximation of $P(E_1)$. Let $H(x, u)$ be as defined for Theorem 1 and u solve

$$p_0 H(\gamma_*(u), u) e^{\sigma u} = b,$$

where $\gamma_*(u) = u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. For the rest of the proof, we will show that

$$P(E_1) = O(1) e^{-\frac{u^2}{2} + O(u^{\varepsilon})}. \tag{33}$$

for any $\varepsilon > 0$. According to the discussion in Section 2, there exists an $\varepsilon_0 > 0$ such that $u > u_h + \varepsilon_0$ and thus $e^{-\frac{u^2}{2} + O(u^{\varepsilon})} = o(1) u_h^{-1} e^{-u_h^{-2}/2}$. If the above bound in (33) can be established, then we can conclude the proof.

First, we derive an approximation for

$$\alpha(u, \varepsilon) = P\Big(\max_{x \in [\frac{L}{2} - u^{-1/2+\varepsilon}, \frac{L}{2} + u^{-1/2+\varepsilon}]} |v'(x)| > b\Big),$$

where $\varepsilon > 0$ is chosen small enough. Then, we split the region $[0, L]$ into $N = \frac{L}{2u^{-1/2+\varepsilon}}$ many intervals each of which is a location shift of $[0, 2u^{-1/2+\varepsilon}]$, i.e. $[2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon} + 2u^{-1/2+\varepsilon}]$. Thanks to the homogeneity of $\xi(x)$, the approximations for

$$P\left(\max_{x \in [2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon} + 2u^{-1/2+\varepsilon}]} |v'(x)| > b\right)$$

are the same for all $1 \le k \le N - 2$. Then, we have

$$P\left(\cup_{k=1}^{N-2} \{\max_{x \in [2ku^{-1/2+\varepsilon}, 2ku^{-1/2+\varepsilon} + 2u^{-1/2+\varepsilon}]} |v'(x)| > b\}\right) \le (1 + o(1)) \frac{L}{2u^{-1/2+\varepsilon}} \alpha(u, \varepsilon).$$

In what follows, we derive an approximation for $\alpha(u, \varepsilon)$. The derivation is similar to the proof of the Theorem 1. Therefore, we omit the details and only lay out the key steps and the major differences. We expand $\xi(x)$ around $x = \frac{L}{2}$ conditional on (by redefining the notations)

$$\xi(\frac{L}{2}) = u + w, \quad \xi'(\frac{L}{2}) = y, \quad \xi''(\frac{L}{2}) = -\Delta(u - z)$$

and obtain that

$$\xi(x) = u + w + \frac{y^2}{2\Delta(u-z)} - \frac{\Delta(u-z)}{2}\left(x - \frac{y}{\Delta(u-z)}\right)^2$$
$$- \frac{Ay}{6\Delta}x^3 + \frac{Au}{24}x^4 + g(x - \frac{L}{2}) + \zeta(x - \frac{L}{2}).$$

Similarly, we have the following proposition for localization.

Proposition 4 *For $\delta' > 3\varepsilon$, let*

$$\mathcal{G}_u = \{|w| > u^{3\varepsilon}\} \cup \{|y| > u^{1/2+4\varepsilon}\} \cup \{|z| > u^{1/2+4\varepsilon}\}$$
$$\cup \Big\{ \sup_{x \notin [-u^{-1/2+\varepsilon}, u^{-1/2+\varepsilon}]} |g(x)| - \delta' u x^2 > 0 \Big\} \cup \Big\{ \sup_{x \in [-u^{-1/2+\varepsilon}, u^{-1/2+\varepsilon}]} |g(x)| > u^{-1/2+\delta'} \Big\}$$

Under the conditions of Theorem 2, we have

$$P(\mathcal{G}_u; \max_{x \in [\frac{L}{2} - u^{-1/2+\varepsilon}, \frac{L}{2} + u^{-1/2+\varepsilon}]} |v'(x)| > b) = o(1) e^{-u^2/2}.$$

Let

$$\mathcal{L}_u = \mathcal{G}_u^c.$$

We now proceed to the factor

$$F(x) - \frac{\int_0^L F(t) e^{\sigma\xi(t)} dt}{\int_0^L e^{\sigma\xi(t)} dt}.$$

Following exactly the same derivation as Lemma 2 in Section 3.1 and noting that $p(x) \equiv p_0$, we have that

$$F(x) - \frac{\int_0^L F(t) e^{\sigma\xi(t)} dt}{\int_0^L e^{\sigma\xi(t)} dt} = p_0 \gamma \exp\left\{ \frac{Ay^3}{3\Delta^4 (u-z)^3 \gamma} + o(u^{-1}) + \omega(u) \right\},$$

where we redefine a change of variable

$$\gamma = x - \frac{L}{2} - \frac{y}{\Delta(u-z)}.$$

Thus, similar to (19), we obtain that

$$v'(x) = e^{\sigma\xi(x)} \left[F(t) - \frac{\int_0^L F(t) e^{\sigma\xi(t)} dt}{\int_0^L e^{\sigma\xi(t)} dt} \right]$$
$$= e^{\sigma u + \sigma w + \frac{\sigma y^2}{2\Delta(u-z)}} \times p_0 \gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2}$$
$$\times \exp\Big\{ \frac{\sigma\Delta z}{2}\gamma^2 - \frac{\sigma A}{6\Delta} y (\gamma + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24} (\gamma + \frac{y}{\Delta(u-z)})^4$$
$$+ \frac{Ay^3}{3\Delta^4 (u-z)^3 \gamma} + o(u^{-1}) + \omega(u) \Big\}.$$

We further simplify the above display and obain that

$$
\begin{aligned}
v'(x) &= e^{\sigma u+\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}} \times p_0\gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2} \\
&\quad \times \exp\Big\{\frac{\sigma\Delta z}{2}\gamma^2 - \frac{\sigma A\gamma^2}{4\Delta^2 u}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + y^3\Big[\frac{\sigma A\gamma}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma}\Big] \\
&\qquad + O(u^{-1}) + \omega(u)\Big\}.
\end{aligned}
$$

For all $|y| \le (1+\varepsilon')\Delta u^{1/2+\varepsilon}$, we have that

$$
\begin{aligned}
\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]} v'(x) &\le \max_{x\in[\frac{L}{2}-(1+2\varepsilon')u^{-1/2+\varepsilon},\frac{L}{2}+(1+2\varepsilon')u^{-1/2+\varepsilon}]} v'(x) \\
&= e^{\sigma u+\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}} \times p_0\gamma_* e^{-\frac{\sigma\Delta u}{2}\gamma_*^2} \\
&\quad \times \exp\Big\{\frac{\sigma\Delta z}{2}\gamma_*^2 - \frac{\sigma A\gamma_*^2}{4\Delta^2 u}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + y^3\Big[\frac{\sigma A\gamma_*}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma_*}\Big] \\
&\qquad + O(u^{-1}+z^2u^{-2}) + \omega(u)\Big\}. \qquad (34)
\end{aligned}
$$

That is, $v'(x)$ is maximized when $x = \frac{L}{2} + \gamma_* + \frac{y}{\Delta(u-z)} + o(u^{-1}) + O(z\gamma_*/u)$. Since $\gamma_* = \Delta^{-1/2}\sigma^{-1/2}u^{-1/2}$, then

$$
\frac{\sigma A\gamma_*}{3\Delta^3 u^2} - \frac{A}{3\Delta^4 u^3\gamma_*} = 0.
$$

Thus, we have that

$$
\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]} v'(x) \quad > \quad b
$$

implies that

$$
\begin{aligned}
\mathcal{A} &\triangleq \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{z}{2u} - \frac{A}{4\Delta^3 u^2}y^2 - \frac{\sigma A}{8\Delta^4 u^3}y^4 + O(z^2u^{-2}) + O(u^{-1}) \\
&\ge \omega(u).
\end{aligned}
$$

Corresponding to the analysis in Section 3.1, the next step is to insert $\mathcal{A}$ to $S(w,y,z)$ and obtain that

$$\begin{aligned}
S(w,y,z) &= u^2+w^2+\frac{\Delta^2(w+z)^2}{A-\Delta^2}+2u(w+\frac{y^2}{2\Delta u})\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{\Delta^2}{A}z^2\\
&\quad +2u\frac{\mathcal{A}}{\sigma}-\frac{y^2z}{\Delta u}-\frac{z}{\sigma}+\frac{A}{2\Delta^3\sigma}\frac{y^2}{u}+\frac{A}{4\Delta^4}\frac{y^4}{u^2}+O(z^2/u)+O(1)\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{2u\mathcal{A}}{\sigma}\\
&\quad +\frac{\Delta^2}{A}z^2-z\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)+\frac{A}{4\Delta^2}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)^2-\frac{A}{4\Delta^2\sigma^2}+O(z^2/u)+O(1)\\
&= u^2+\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}+\frac{2u\mathcal{A}}{\sigma}\\
&\quad +\Big[\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)\Big]^2-\frac{A}{4\Delta^2\sigma^2}+O(u^{8\varepsilon}).
\end{aligned}$$

For the last step in the above derivation, we use the fact that, on the set $\mathcal{L}_u$, $O(z^2/u)=O(u^{8\varepsilon})$. Thus,

$$\begin{aligned}
&P\left(\max_{x\in[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|v'(x)|>b\right)\\
&= \Delta\int_{\mathcal{L}_u}h(w,y,z)P(\max_{x\in[-u^{-1/2+\varepsilon},u^{-1/2+\varepsilon}]}|v'(x)|>b|w,y,z)dwdydz\\
&= O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})+\frac{A}{8\Delta^2\sigma^2}}\int_{\mathcal{L}_u}P(\mathcal{A}>\omega(u))\\
&\quad\times\exp\Big\{-\frac{u\mathcal{A}}{\sigma}-\frac{1}{2}\frac{(\sqrt{A}w+\Delta^2A^{-1/2}z)^2}{A-\Delta^2}-\frac{1}{2}\Big[\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}\Big(\frac{y^2}{\Delta u}+\frac{1}{\sigma}\Big)\Big]^2\Big\}dwdydz.
\end{aligned}$$

We introduce a change of variable

$$B=\frac{\Delta z}{\sqrt{A}}-\frac{\sqrt{A}}{2\Delta}(\frac{y^2}{\Delta u}+\frac{1}{\sigma}).$$

Then,

$$\begin{aligned}
\sqrt{A}w+\Delta^2A^{-1/2}z &= \Delta B+\sqrt{A}w+\frac{\sqrt{A}}{2}(\frac{y^2}{\Delta u}+\frac{1}{\sigma})\\
&= \frac{\sqrt{A}}{2\sigma}+\Delta B+\sqrt{A}\mathcal{A}+o(1).
\end{aligned}$$

Thus, by dominated convergence theorem and applying the change of variable from (w,z,y) to $(\mathcal{A},B,y)$, we have that

$$P\left(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b;|y|\le(1+\varepsilon')\Delta u^{1/2+\varepsilon};\mathcal{L}_u\right)=O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}. \quad (35)$$

For $|y|>(1+\varepsilon')\Delta u^{1/2+\varepsilon}$, note that the function $|v'(x)|$ is maximized at $x=\frac{L}{2}+\gamma_*+\frac{y}{\Delta(u-z)}$, that is outside the interval $[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]$. Therefore, $\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|$ is less than the estimate in (34) by at least a factor of $e^{-\lambda u^{2\varepsilon}}$ (by considering the dominating term $\gamma e^{-\frac{\sigma\Delta u}{2}\gamma^2}$). Therefore,

$$\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b$$

if

$$\mathcal{A}=\sigma w+\frac{\sigma y^2}{2\Delta(u-z)}+\frac{z}{2u}-\frac{A}{4\Delta^3u^2}y^2-\frac{\sigma A}{8\Delta^4u^3}y^4+O(z^2/u^2)+O(u^{-1})>\lambda u^{2\varepsilon}+\omega(u).$$

Thus,

$$\begin{aligned}&P\Big(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b;(1-\varepsilon')\Delta u^{1/2+\varepsilon}\leq|y|\leq u^{1/2+4\varepsilon};\mathcal{L}_u\Big) \qquad (36)\\ =\;& O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.\end{aligned}$$

We combine the solution of (35), (36), Lemma 4 and obtain that

$$\begin{aligned}\alpha(u,\varepsilon)&=P\left(\max_{x\in[\frac{L}{2}-u^{-1/2+\varepsilon},\frac{L}{2}+u^{-1/2+\varepsilon}]}|v'(x)|>b\right)\\&=O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.\end{aligned}$$

Thus

$$P(E_1)=O(1)u^{1/2-\varepsilon}\alpha(u,\varepsilon)=O(1)e^{-\frac{u^2}{2}+O(u^{8\varepsilon})}.$$

As ε can be chosen arbitrarily small, we obtain (33) by redefining ε.

B Proofs of Propositions

Proof of Proposition 1. The proof needs a change of measure described as follows. For $\zeta\in R$, let

$$A_\zeta=\{x:\xi(x)>\zeta\}\cap[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$$

be the excursion set (on the interval $[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$) over level ζ and let P be the underlying nominal (original) probability measure. Define $Q_\zeta(\cdot)$ via

$$dQ_\zeta=\frac{mes(A_\zeta)}{E(mes(A_\zeta))}dP=\frac{mes(A_\zeta)}{\int_{x_*+u^{-1/2+\delta/2}}^{L-u^{-1/2+\delta}}P(\xi(x)>\zeta)dx}dP, \qquad (37)$$

where $E(\cdot)$ is the expectation under P and $mes(A_\zeta)$ is the Lebesgue measure of the excursion set above level ζ. Note that under Q_ζ, almost surely $\sup_L\xi(x)>\zeta$. In order to generate sample paths according Q_ζ, one first simulates τ with density function $\{h(\tau):\tau\in[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]\}$

$$h(\tau)=\frac{P(\xi(\tau)>b)}{E(mes(A_\zeta))} \qquad (38)$$

that is a uniform distribution over the interval $[x_*+u^{-1/2+\delta/2},L-u^{-1/2+\delta}]$; then simulate $\xi(\tau)$ conditional distribution (under the original law) given that $\xi(\tau)>\zeta$; lastly simulate $\{\xi(x):x\neq\tau\}$

given $(\tau, \xi(\tau))$ according to the original distribution. If ζ is suitably chosen, Q_ζ serves as a good approximation of the conditional distribution of $\xi(x)$ given that $\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > b$.

Lemma 10 *Under conditions in Theorem 1, we have that*

$$P\Big(\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2, E_1\Big) = o(u^{-1}e^{-u^2/2}).$$

Proof of Lemma 10. Let

$$F_b = \{\sup_{x\in[x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2\}.$$

Let $\zeta = u - (\log u)^2 - 1/u$. Then, the probability can be written as

$$\begin{aligned} P(F_b, E_1) &\leq O(1)E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; F_b, E_1\right] \\ &= O(1)\int_{x_*+u^{-1/2+\delta/2}}^{L-u^{-1/2+\delta}} E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; F_b, E_b\right] d\tau, \end{aligned}$$

where we use E_τ^Q to denote the conditional expectation $E^Q(\cdot|\tau)$ under the measure Q_ζ. Given a particular $\tau \in [x_* + u^{-1/2+\delta/2}, L - u^{-1/2+\delta}]$, we redefine the change of variables

$$\xi(\tau) = u + w, \xi'(\tau) = y, \xi''(\tau) = -\Delta(u - z).$$

Note that the current definition of (w, y, z) is different from that in the proposition and Theorem 1. As the previous definition of (w, y, z) will not be used in this lemma, to simplify the notation, we do not create another notation and use (w, y, z) differently. Conditional on (w, y, z) the process $g(x)$ is a mean zero Gaussian process such that

$$\xi(x) = E(\xi(x)|w, y, z) + g(x - \tau).$$

We have the bound of the excursion set that $E^Q(1/mes(A_\zeta)) = O(u)$, the detailed development of which is omitted. With this in mind, we first have that that

$$E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |z| \geq u^{1/2+\delta/16}, F_b, E_b\right] = o(u^{-1}e^{-u^2/2}).$$

and similarly

$$E^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y| \geq u^{1/2+\delta/16}, F_b, E_b\right] = o(u^{-1}e^{-u^2/2}).$$

In addition, for some λ_0 sufficiently large and δ_0 small, we have that

$$\begin{aligned} & E\left(\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; \sup_{|x|\leq u^{-1/2+\delta}} |g(x)| > \lambda_0 u^{-1+4\delta}, \text{ or } \sup_{|x|>u^{-1/2+\delta}} |g(x)| - \delta_0 u x^2 > 0\right) \\ = & \ o(u^{-1}e^{-u^2/2}). \end{aligned}$$

Then, we only need to consider the situation that $|y| < u^{1/2+\delta/16}$ and $|z| < u^{1/2+\delta/16}$. Furthermore, using Taylor expansion on $\xi(x)$ as we had done several times previously, the process $\xi(x)$ is a approximately a quadratic function with mode being $\tau+\frac{y}{\Delta\sigma(u-z)}$ for $\tau \in [x_*+u^{-1/2+\delta/2}, L-u^{-1/2+\delta}]$. Thus, when considering the integral $\int_0^L e^{\xi(t)}dt$ and $\int_0^L (F(x)-F(t))e^{\xi(t)}dt$, we do not have to consider the boundary issue as in the analysis of $P(E_2)$. With the same calculations for (21) by expanding ξ at τ instead of x_*, we obtain that

$$\sup_{x\in[u^{-1/2+\delta}, L-u^{-1/2+\delta}]} |v'(x)| \geq b$$

if and only if

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta(u-z)} + \frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad -\frac{\sigma A}{6\Delta}y(\gamma_* + \frac{y}{\Delta(u-z)})^3 + \frac{\sigma A u}{24}(\gamma_* + \frac{y}{\Delta(u-z)})^4 \\
&\quad -\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) \\
&\quad -\frac{Ay^3}{\Delta^4(u-z)^3\gamma_*} + \log\frac{p(x)}{p(x_*)} \\
&\geq o(u^{-1}) + \omega(u),
\end{aligned}$$

where the x in "$p(x)$" is $x = \tau + \gamma_* + \frac{y}{\Delta(u-z)} + o(u^{-1}) + O(z\gamma_*/u)$. Similar to the derivation for (39), we expand the second row in the definition of $\mathcal{A}$ and obtain that

$$\begin{aligned}
\mathcal{A} &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2}y^2 z - \frac{\sigma A y^4}{8\Delta^4(u-z)^3} + \frac{\sigma\Delta z}{2}\gamma_*^2 - \frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2 + \frac{\sigma A(u-z)}{24}\gamma_*^4 \\
&\quad -\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x_*)}{6p(x_*)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) + \log\frac{p(x)}{p(x_*)}.
\end{aligned}$$

Notice that

$$\frac{p''(x)}{6p(x)}(\gamma_*^2 + \frac{3}{\sigma\Delta(u-z)}) = O(u^{-1}).$$

When $|x - x_*| < \varepsilon$, by Taylor expansion

$$\left|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})\right| = O((x-x_*)/\sqrt{u}) = o(\log p(x) - \log p(x_*));$$

when $|x - x_*| > \varepsilon$

$$\left|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})\right| = O(u^{-1/2}) = o(1) = o(\log p(x) - \log p(x_*)).$$

Therefore $\left|\frac{p'(x)}{2p(x)\gamma_*}(\gamma_*^2 + \frac{1}{\sigma\Delta(u-z)})\right|$ is always of a smaller order than $\log p(x) - \log p(x_*)$. On the region $|x - x_*| > \frac{u^{-1/2+\delta/2}}{2}$, there exists a positive λ such that

$$\log\frac{p(x)}{p(x_*)} \leq -2\lambda u^{-1+\delta}.$$

Thus, $\mathcal{A}$ is bounded by

$$\begin{aligned}\mathcal{A} < \mathcal{A}' &= \sigma w + \frac{\sigma y^2}{2\Delta u} + \frac{\sigma}{2\Delta u^2} y^2 z - \frac{\sigma A y^4}{8\Delta^4 (u-z)^3} + \frac{\sigma \Delta z}{2}\gamma_*^2 - \frac{\sigma A y^2}{4\Delta^2 (u-z)}\gamma_*^2 + \frac{\sigma A(u-z)}{24}\gamma_*^4 \\ &\quad -\lambda u^{-1+\delta}\end{aligned}$$

Furthermore, notice that

$$\begin{aligned}& E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y|, |z| \le u^{1/2+\delta/16}, F_b, E_1\right] \\ \le\ & O(1)\int_{w \ge -(\log u)^2} e^{-\frac{1}{2}S(w,y,z)} \frac{P(\mathcal{A}' \ge \omega(u), F_b)}{mes(A_\zeta)} dwdydz.\end{aligned}$$

Similar to the previous development, we write

$$\begin{aligned}S(w,y,z) &= u^2 + w^2 + \frac{\Delta^2 (w-z)^2}{A - \Delta^2} + 2u(w + \frac{y^2}{2\Delta u}) \\ &= u^2 + w^2 + \frac{\Delta^2 (w-z)^2}{A - \Delta^2} \\ &\quad + 2u\Big[\frac{\mathcal{A}'}{\sigma} - \frac{y^2 z}{2\Delta u^2} + \frac{A y^4}{8\Delta^4 (u-z)^3} - \frac{\Delta z}{2}\gamma_*^2 + \frac{A y^2}{4\Delta^2 (u-z)}\gamma_*^2 - \frac{A(u-z)}{24}\gamma_*^4 + \lambda u^{-1+\delta}/\sigma\Big].\end{aligned}$$

Thus, by dominated convergence theorem and the fact that $mes(A_\zeta)^{-1} = O(u)$, we have that

$$\begin{aligned}& E_\tau^Q\left[\frac{P(Z > u - (\log u)^2)}{mes(A_\zeta)}; |y|, |z| \le u^{1/2+\delta/16}, F_b, E_1\right] \\ \le\ & O(1)\int_{|y|,|z| \le u^{-1/2+\varepsilon/4}} E(mes(A_\zeta)^{-1}; \mathcal{A}' \ge \omega(u)) e^{-\frac{1}{2}S(w,y,z)} dwdydz \\ \le\ & O(1) e^{-\frac{u^2}{2} - \lambda u^\delta/\sigma} \\ & \times \int_{|y|,|z| \le u^{-1/2+\varepsilon/4}} E(mes(A_\zeta)^{-1}; \mathcal{A}' \ge \omega(u)) \\ & \qquad \times \exp\left[-\frac{\Delta^2}{2(A - \Delta^2)} z^2 - \frac{u\mathcal{A}'}{\sigma} + \frac{y^2 z}{2\Delta u} - \frac{A y^4}{8\Delta^4 u^2} + \frac{z}{2\sigma} + \frac{A y^2}{4\Delta^3 \sigma u}\right] dwdydz \\ =\ & o(u^{-1} e^{-u^2/2}).\end{aligned}$$

■

With a completely analogous proof as the Lemma 10, we have that

Lemma 11 *Under conditions in Theorem 1, we have that*

$$P\left(\sup_{x \in [u^{-1/2+\delta}, x_* - u^{-1/2+\delta/2}]} \xi(x) > u - (\log u)^2, E_1\right) = o(u^{-1} e^{-u^2/2}).$$

We write

$$J_b = \{\sup_{x \in [u^{-1/2+\delta}, x_* - u^{-1/2+\delta/2}]} \xi(x) > u - (\log u)^2\} \cup \{\sup_{x \in [\tau_* + u^{-1/2+\delta/2}, L - u^{-1/2+\delta}]} \xi(x) > u - (\log u)^2\}$$

and thus

$$P(J_b^c, E_1) = o(u^{-1}e^{-u^2/2}).$$

We proceed to the following lemma to complete the proof of the proposition.

Lemma 12 *Let (w, y, z) defined as in Section 3.1. For $\varepsilon > 0$, let*

$$L_b = \{|w| < u^{3\delta}, |y| < u^{1/2+4\delta}, |z| < u^{1/2+4\delta}\}$$

Under conditions of Theorem 1, we have that

$$P\left(L_b^c, J_b^c, E_1\right) = o(u^{-1}e^{-u^2/2}).$$

Proof. Note that $|v'(x)| > b$ implies that $\xi(x) > \log b - \kappa_0 = u - O(\log u)$ for some $\kappa_0 > 0$. Thus, on the set J_b^c, E_1 implies that $\sup_{[x_* - u^{-1/2+\delta/2}, x_* + u^{-1/2+\delta/2}]} \xi(x) > \frac{\log b}{\sigma} - (\log u)^2$. Therefore, we have that

$$P(|w| > u^{3\delta}, F_b^c, E_b) \leq P(|w| > u^{3\delta}, \sup_{[x_* - u^{-1/2+\delta/2}, x_* + u^{-1/2+\delta/2}]} \xi(x) > \frac{\log b}{\sigma} - (\log u)^2) = o(u^{-1}e^{-u^2/2}),$$

where the last step is an application of Borel-TIS lemma ([6, 15, 4]). Furthermore, by simply bound of Gaussian distribution, we have that

$$P(|w| < u^{3\delta}, |z| > u^{1/2+4\delta}, F_b^c, E_b) = o(u^{-1}e^{-u^2/2}),$$

and

$$P(|w| < u^{3\delta}, |y| > u^{1/2+4\delta}, F_b^c, E_b) = o(u^{-1}e^{-u^2/2}).$$

We thus conclude the proof. ∎

The results of Lemmas 10, 11, and 12 immediately lead to the conclusion of Proposition 1. ∎

Proof of Proposition 2. Note that $g(x)$ is independent of (w, y, z) and $\mathcal{L}_u$ only depends on (w, y, z). Therefore,

$$\begin{aligned} P\left(\sup_{|x| > u^{-1/2+8\delta}} [|g(x)| - \delta' u x^2] > 0,\ \mathcal{L}_u\right) &= P\left(\sup_{|x| > u^{-1/2+8\delta}} [|g(x)| - \delta' u x^2] > 0\right) P(\mathcal{L}_u) \\ &= o(u^{-1}e^{-u^2/2}). \end{aligned}$$

The last step is a direct application of the Borel-TIS lemma and the fact that $P(\mathcal{L}_u) = O(e^{-u^2/2 + O(u^{1+3\delta})})$. With a similar argument, we obtain the second bound. ∎

Proof of Propositions 3 and 4. The proofs of these two propositions are completely analogous to that of Proposition 1, that is, basically a repeated application of Borel-TIS lemma and the change of measure Q_ζ. Therefore, we omit the details. ∎

C Proof of the Lemmas

Proof of Lemma 1. On the set $|x - x_*| < u^{-1/2+8\delta}$ and $\mathcal{L}'_u$, we have $s = O(u^{8\delta})$ and thus

$$\frac{y^3 s}{(u-z)^{5/2}} = O(u^{-1+20\delta}), \frac{y^2 s^2}{(u-z)^2} = O(u^{-1+24\delta}), \frac{s^4}{(u-z)} = O(u^{-1+32\delta}).$$

Let X be a standard Gaussian random variable. We conclude the proof by the following calcuation

$$
\begin{aligned}
&\int_{|x-x_*|<u^{-1/2+8\delta}} e^{\sigma[-\frac{s^2}{2}-\frac{Ay^3}{\Delta^{7/2}(u-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{A}{24\Delta^2(u-z)}s^4]}ds \\
= \;& e^{o(u^{-1})}\int_{|x-x_*|<u^{-1/2+8\delta}} e^{-\frac{\sigma s^2}{2}}\times\left(1-\frac{\sigma Ay^3}{\Delta^{7/2}(u-z)^{5/2}}s-\frac{\sigma Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{\sigma A}{24\Delta^2(u-z)}s^4\right)ds \\
= \;& e^{o(u^{-1})}\sqrt{\frac{2\pi}{\sigma}}E\left[1-\frac{A\sigma^{1/2}y^3X}{\Delta^{7/2}(u-z)^{5/2}}-\frac{Ay^2X^2}{4\Delta^3(u-z)^2}+\frac{AX^4}{24\Delta^2\sigma(u-z)}\right] \\
= \;& \sqrt{\frac{2\pi}{\sigma}}\exp\left\{-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma(u-z)}+o(u^{-1})\right\} \\
= \;& \sqrt{\frac{2\pi}{\sigma}}\exp\left\{-\frac{Ay^2}{4\Delta^3(u-z)^2}+\frac{A}{8\Delta^2\sigma u}+o(u^{-1})\right\}.
\end{aligned}
$$

■

Proof of Lemma 2. We use the result of Lemma 1 and the Taylor expansion

$$F(x)-F(t)=p(x)(x-t)-\frac{1}{2}p'(x)(x-t)^2+\frac{1}{6}p''(x)(x-t)^3+o(x-t)^4.$$

Recall the change of variable

$$s(t)=\sqrt{\Delta(u-z)}\left(t-x_*-\frac{y}{\Delta(u-z)}\right)$$

at the beginning of Step 1 of the main proof. We apply it to the spatial index t. Note that $t-x_*-s(t)/\sqrt{\Delta(u-z)}=y/(\Delta(u-z))$ and $x-t=\gamma-s(t)/\sqrt{\Delta(u-z)}$. We perform the same splitting as in (15), insert the result in (16), use the expansion of ξ in (13), and obtain that

$$
\begin{aligned}
&\left(\int_0^L e^{\sigma\xi(t)}dt\right)^{-1}\int_0^L (F(x)-F(t))e^{\sigma\xi(t)}dt \\
= \;& \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2}-\frac{A}{8\Delta^2\sigma(u-z)}+\omega(u)+o(u^{-1})\right\} \\
&\times\int_{|s|\le u^{8\delta}}\left[p(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)-\frac{1}{2}p'(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)^2\right. \\
&\qquad\left.+\frac{1}{6}p''(x)\Big(\gamma-\frac{s}{\sqrt{\Delta(u-z)}}\Big)^3+o(u^{-3/2})\right] \\
&\quad\times\sqrt{\frac{\sigma}{2\pi}}e^{\sigma[-\frac{s^2}{2}-\frac{Ay^3}{3\Delta^{7/2}(u-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u-z)^2}s^2+\frac{A}{24\Delta^2(u-z)}s^4]}ds
\end{aligned}
$$

We rewrite the above integral by pulling out the Gaussian density and expanding the exponential term in the last row

$$
\begin{aligned}
&= \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
&\quad \times \int_{|s|\le u^{8\delta}|} \sqrt{\frac{\sigma}{2\pi}} e^{-\frac{\sigma s^2}{2}} \\
&\quad \times \left[p(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big) - \frac{1}{2}p'(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big)^2 + \frac{1}{6}p''(x)\Big(\gamma - \frac{s}{\sqrt{\Delta(u-z)}}\Big)^3\right] \\
&\quad \times \left[1 - \frac{\sigma A y^3}{3\Delta^{7/2}(u-z)^{5/2}}s - \frac{\sigma A y^2}{4\Delta^3(u-z)^2}s^2 + \frac{\sigma A}{24\Delta^2(u-z)}s^4\right] ds.
\end{aligned}
$$

Similar to Lemma 1, we further evaluate the above integral by computing moments of $N(0, \sigma^{-1/2})$ and obtain that (we omit several cross terms that can be absorbed by $o(u^{-1})$)

$$
\begin{aligned}
& F(x) - \frac{\int_0^L F(t)e^{\sigma\xi(t)}dt}{\int_0^L e^{\sigma\xi(t)}dt} \\
= \ & \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
& \times \left[p(x)\gamma - \frac{p'(x)}{2}\left(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}\right) + \frac{p''(x)}{6}\left(\gamma^3 + \frac{3\gamma}{\sigma\Delta(u-z)}\right)\right. \\
& \qquad \left. + p(x)\frac{Ay^3}{3\Delta^4(u-z)^3} - p(x)\gamma\frac{Ay^2}{4\Delta^3(u-z)^2} + p(x)\gamma\frac{A}{8\sigma\Delta^2(u-z)}\right].
\end{aligned}
$$

We take out the factor "$p(x)\gamma$" from the bracket and continue the calculation

$$
\begin{aligned}
= \ & \exp\left\{\frac{Ay^2}{4\Delta^3(u-z)^2} - \frac{A}{8\Delta^2\sigma(u-z)} + \omega(u) + o(u^{-1})\right\} \\
& \times p(x)\gamma \exp\Big[-\frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\
& \qquad + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} - \frac{Ay^2}{4\Delta^3(u-z)^2} + \frac{A}{8\sigma\Delta^2(u-z)}\Big].
\end{aligned}
$$

We further simplify the above display and obtain that

$$
\begin{aligned}
= \ & p(x)\gamma \exp\Big[-\frac{p'(x)}{2p(x)\gamma}(\gamma^2 + \frac{1}{\sigma\Delta(u-z)}) + \frac{p''(x)}{6p(x)}(\gamma^2 + \frac{3}{\sigma\Delta(u-z)}) \\
& \qquad + \frac{Ay^3}{3\Delta^4(u-z)^3\gamma} + o(u^{-1}) + \omega(u)\Big].
\end{aligned}
$$

■

Proof of Lemma 3. Let $\mathcal{A}$ be defined as in (21). Note that $p'(x_*) = 0$ and $p'(x) \sim p''(x_*)(\gamma + y/\Delta(u-z))$. We apply Taylor expansion of the term $\log\frac{p(x_*+\gamma_*+\Delta^{-1}(u-z)^{-1}y)}{p(x_*)}$ in (21) and expand the second row of (21). Thus, $\mathcal{A}$ can be further simplified to

$$\begin{aligned}
\mathcal{A} &= \sigma w+\frac{\sigma y^2}{2\Delta u}+\frac{\sigma}{2\Delta u^2}y^2 z-\frac{\sigma A y^4}{8\Delta^4(u-z)^3}+\frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad -\frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_*-\frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2+\frac{\sigma A u}{24}\gamma_*^4 \\
&\quad -\frac{p''(x_*)}{2p(x_*)}(\gamma_*+\frac{y}{\Delta(u-z)})(\gamma_*+\frac{1}{\sigma\Delta(u-z)\gamma_*}) \\
&\quad +\frac{p''(x_*)}{6p(x_*)}(\gamma_*^2+\frac{3}{\sigma\Delta(u-z)})+\frac{A y^3}{3\Delta^4(u-z)^3\gamma_*} \\
&\quad +\frac{p''(x_*)}{2p(x_*)}(\gamma_*+\frac{y}{\Delta(u-z)})^2+o(y^2u^{-2})+O(z^2/u^2).
\end{aligned}$$

Note that $\gamma_*=u^{-1/2}\Delta^{-1/2}\sigma^{-1/2}$. The term

$$-\frac{p''(x_*)}{2p(x_*)}\frac{y}{\Delta(u-z)}(\gamma_*+\frac{1}{\sigma\Delta(u-z)\gamma_*})$$

expanded from the third row cancels the cross term

$$\frac{\gamma_* p''(x_*)}{p(x_*)}\frac{y}{\Delta(u-z)}$$

expanded from the quadratic term in the last row. Then, $\mathcal{A}$ is further simplified to

$$\begin{aligned}
\mathcal{A} &= \sigma w+\frac{\sigma y^2}{2\Delta u}+\frac{\sigma}{2\Delta u^2}y^2 z-\frac{\sigma A y^4}{8\Delta^4(u-z)^3}+\frac{\sigma\Delta z}{2}\gamma_*^2 \\
&\quad -\frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_*-\frac{\sigma A y^2}{4\Delta^2(u-z)}\gamma_*^2+\frac{\sigma A (u-z)}{24}\gamma_*^4 \\
&\quad -\frac{p''(x_*)}{2p(x_*)}(\gamma_*^2+\frac{1}{\sigma\Delta(u-z)}) \\
&\quad +\frac{p''(x_*)}{6p(x_*)}(\gamma_*^2+\frac{3}{\sigma\Delta(u-z)})+\frac{A y^3}{3\Delta^4(u-z)^3\gamma_*} \\
&\quad +\frac{p''(x_*)}{2p(x_*)}(\gamma_*^2+\frac{y^2}{\Delta^2(u-z)^2})+o(y^2u^{-2})+O(z^2/u^2).
\end{aligned} \tag{39}$$

Furthermore, the term $-\frac{\sigma A y^3}{3\Delta^3(u-z)^2}\gamma_*$ in the second row cancels $\frac{A y^3}{3\Delta^4(u-z)^3\gamma_*}$ in the fourth row. We now plug in $\gamma_*^2=\Delta^{-1}\sigma^{-1}u^{-1}$ and obtain that

$$\begin{aligned}
\mathcal{A} &= \sigma w+\frac{\sigma y^2}{2\Delta u}+\frac{\sigma}{2\Delta u^2}y^2 z-\frac{\sigma A y^4}{8\Delta^4 u^3}+\frac{z}{2u} \\
&\quad -\frac{A y^2}{4\Delta^3 u^2}+\frac{A}{24\sigma\Delta^2 u}-\frac{p''(x_*)}{3p(x_*)\sigma\Delta u}+\frac{p''(x_*)}{2p(x_*)}(\frac{1}{\sigma\Delta u}+\frac{y^2}{\Delta^2 u^2})+o(u^{-1})+O(z^2/u) \\
&= \sigma w+\frac{\sigma y^2}{2\Delta u}+\frac{\sigma}{2\Delta u^2}y^2 z+\frac{z}{2u}+\frac{A}{24\sigma\Delta^2 u}+\frac{p''(x_*)}{6p(x_*)\sigma\Delta u} \\
&\quad -\frac{\sigma A y^4}{8\Delta u^3}+\frac{y^2}{u^2}(-\frac{A}{4\Delta^3}+\frac{p''(x_*)}{2p(x_*)\Delta^2})+o(u^{-1}+y^2u^{-2})+O(z^2/u^2).
\end{aligned}$$

■

Proof of Lemma 5. By simple algebra, we have that

$$
\begin{aligned}
S(w,y,z) &= u^2 + 2u\mathcal{A}/\sigma + \frac{(\sqrt{A}w + \Delta^2 A^{-1/2} z)^2}{A-\Delta^2} + \frac{\Delta^2}{A} z^2 \\
&\quad - \frac{y^2 z}{\Delta u} - \frac{z}{\sigma} + \frac{A}{4\Delta^4 u^2} y^4 - \frac{y^2}{u}\Big(-\frac{A}{2\sigma\Delta^3} + \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\Big) + o(y^2/u) + O(z^2/u) + O(1) \\
&= u^2 + 2u\mathcal{A}/\sigma + \frac{(\sqrt{A}w + \Delta^2 A^{-1/2} z)^2}{A-\Delta^2} + \frac{\Delta^2}{A}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big)^2 \\
&\quad + \frac{1}{\sigma}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big) - \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\frac{y^2}{u} + o(y^2/u) + O(z^2/u) + O(1).
\end{aligned}
$$

Note that, on the set $\mathcal{L}'_u$, $o(y^2/u) + O(z^2/u) = o(y^2/u + z)$ and thus,

$$
S(w,y,z) \geq u^2 + 2u\mathcal{A}/\sigma + \frac{\Delta^2}{A}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big)^2 + \frac{1+o(1)}{\sigma}\Big(\frac{A}{2\Delta^3}\frac{y^2}{u} - z\Big) - \frac{p''(x_*)}{p(x_*)\sigma\Delta^2}\frac{y^2}{u} + O(1).
$$

■

Proof of Lemma 6. Using the second change of variable in (24), the denominator in (25) is

$$
\int_0^L e^{\sigma\xi(t)}dt = e^{c_*}\int_0^L \exp\Big\{\sigma\Big[-\frac{s^2}{2} - \frac{Ay^3}{3\Delta^{7/2}u_L^{5/2}}s - \frac{Ay^2}{4\Delta^3 u_L^2}s^2 + \frac{A}{24\Delta^2 u_L}s^4\Big]\Big\}dt.
$$

Let Z be a standard Gaussian random variable following $N(0,1)$. With a similar splitting in (15) and the derivation in Lemma 1 and noticing the boundary constraint that

$$
t \leq L \Longleftrightarrow s \leq \sqrt{\frac{(1-z/u_L)}{\sigma}}\zeta_L - \frac{y}{\sqrt{\Delta(u_L - z)}},
$$

we apply Taylor expansion on the integrand and have that

$$
\begin{aligned}
&= \frac{\sqrt{2\pi}e^{c_* + o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L - z)}} e^{\omega(u_L)} \\
&\quad \times E\left[1 - \frac{\sigma^{1/2}Ay^3}{3\Delta^{7/2}u_L^{5/2}}Z - \frac{Ay^2}{4\Delta^3 u_L^2}Z^2 + \frac{A}{24\Delta^2\sigma u_L}Z^4; Z \leq \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\right] \\
&= \frac{\sqrt{2\pi}e^{c_* + o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L - z)}} e^{\omega(u_L) + O(y^3/u_L^{5/2} + y^2/u_L^2)} \\
&\quad \times E\left[1 + \frac{A}{24\Delta^2\sigma u_L}Z^4; Z \leq \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}}y\right],
\end{aligned}
$$

where $c_* = \sigma(u_L + w + \frac{y^2}{2\Delta(u_L - z)} - \frac{Ay^4}{8\Delta^4(u_L - z)^3})$ and $\omega(u) = O(\sup_{|x|\leq u^{-1/2+8\delta}}|g(x)|)$. The expectation in the previous display can be written as

$$
\begin{aligned}
& E\Big[1+\frac{A}{24\Delta^2\sigma u_L}Z^4; Z\le \sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\\
= \;& P\Big[Z\le \sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\\
& \times\exp\Big\{\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)+\omega(u_L)+O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})\Big\}
\end{aligned}
$$

We use the fact that $E(Z^4|Z\le\zeta_L)=E(Z^4|Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y)+o(1+yu^{-1/2})$. We continue the calculations and obtain that

$$
\begin{aligned}
\int_0^L e^{\sigma\xi(t)}dt \;=\;& \frac{\sqrt{2\pi}e^{c_*+o(u_L^{-1})}}{\sqrt{\Delta\sigma(u_L-z)}}P\Big[Z\le \sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\\
& \times\exp\Big\{\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)+\omega(u_L)+O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})\Big\}.
\end{aligned}
$$

We now proceed to the numerator of (25). Using Taylor expansion

$$
F(x)-F(t)=p(x)(x-t)-\frac{1}{2}p'(x)(x-t)^2+\frac{1}{6}p''(x)(x-t)^3+o(x-t)^3,
$$

the numerator of (25) is (with the splitting as in (15))

$$
\begin{aligned}
& \int_0^L (F(x)-F(t))e^{\sigma\xi(t)}dt\\
= \;& \frac{e^{c_*+\omega(u_L)+o(u_L^{-1})}}{\sqrt{\Delta(u_L-z)}}\times\int_{-u^{8\delta}}^{\sqrt{\frac{(1-z/u)}{\sigma}}\zeta_L-\frac{y}{\sqrt{\Delta(u-z)}}}\\
& \Big[p(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})-\frac{1}{2}p'(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})^2+\frac{1}{6}p''(x)(\gamma-\frac{s}{\sqrt{\Delta(u_L-z)}})^3+o(u_L^{-3/2})\Big]\\
& \times e^{\sigma\left\{-\frac{s^2}{2}-\frac{Ay^3}{3\Delta^{7/2}(u_L-z)^{5/2}}s-\frac{Ay^2}{4\Delta^3(u_L-z)^2}s^2+\frac{A}{24\Delta^2(u_L-z)}s^4\right\}}ds\\
= \;& \sqrt{\frac{2\pi}{\Delta\sigma(u_L-z)}}e^{c_*+\omega(u_L)+o(u_L^{-1})}\\
& \times E\Big\{p(x)(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})-\frac{p'(x)}{2}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^2+\frac{p''(x)}{6}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^3\\
& \quad+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})\\
& \quad+O(y^3/u_L^{5/2}+y^2/u_L^2+u_L^{-2})\;\;;\;\; Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}.
\end{aligned}
$$

Thus, the factor in (25) is

$$\begin{aligned}
&\int_0^L (F(x)-F(t))\frac{e^{\sigma\xi(t)}}{\int_0^L e^{\sigma\xi(s)}ds}dt\\
= &\exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\}\\
&\times E\Big\{p(x)(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})-\frac{p'(x)}{2}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^2+\frac{p''(x)}{6}(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})^3\\
&\quad+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma-\frac{Z}{\sqrt{\Delta\sigma(u_L-z)}})\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}
\end{aligned}$$

where $\lambda(u_L)=O(y^3/u_L^{5/2}+y^2/u_L^2+y/u^{3/2})+o(u_L^{-1}+u_L^{-1}z)$. We take out a factor $\sqrt{\Delta\sigma(u_L-z)}$ from the above expectation and obtain that

$$\begin{aligned}
= &\exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\}\\
&\frac{1}{\sqrt{\Delta\sigma u_L(1-z/u_L)}}E\Big\{p(x)(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)-\frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)^2\\
&+\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big\}.
\end{aligned}$$

Notice that in the last two terms of the above display and for the denominator of the second term in the second low, "u_L-z" is replaced by u_L. The error caused by this change can be absorbed into $\lambda(u_L)$. Notice that

$$\frac{1}{\sqrt{\Delta\sigma u_L(1-z/u_L)}}=\frac{e^{\frac{z}{2u_L}+o(z/u_L)}}{\sqrt{\Delta\sigma u_L}}.$$

We further separate the expectation into two parts and obtain that

$$\begin{aligned}
= &\exp\Big\{-\frac{A}{24\Delta^2\sigma u_L}E\left(Z^4|Z\le\zeta_L\right)+\lambda(u_L)+\omega(u_L)\Big\}\times\frac{e^{\frac{z}{2u_L}}}{\sqrt{\Delta\sigma u_L}}\\
&\times\Big\{E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)\\
&\qquad-\frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L-z)}-Z)^2\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\\
&\quad+E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3\\
&\qquad+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\ \Big|\ Z\le\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big]\Big\}.
\end{aligned}$$

Thus, we conclude the proof. ■

Proof of Lemma 7. Similar to the calculations resulting (18), we obtain that

$$\begin{aligned}
\xi(x) &= u_L + w + \frac{y^2}{2\Delta(u_L - z)} - \frac{\Delta(u_L - z)}{2}\gamma^2 - \frac{A}{6\Delta} y(\gamma + \frac{y}{\Delta(u_L - z)})^3 + \frac{Au_L}{24}(\gamma + \frac{y}{\Delta(u_L - z)})^4 \\
&\quad + g(x - t_L) + \vartheta(x - t_L) \\
&= u_L + w + \frac{y^2}{2\Delta u_L} - \frac{\Delta(u_L - z)}{2}\gamma^2 + \frac{Au_L}{24}\gamma^4 + o(u^{-1}y^2) + g(x - t_L) + \vartheta(x - t_L),
\end{aligned}$$

where $\vartheta(x) = O(u^{1/2+4\delta}x^5 + ux^6)$. Combining the above expression and Lemma 6, we obtain that

$$\begin{aligned}
v'(x) &= e^{\sigma\xi(x)} \int_0^L (F(x) - F(t)) \frac{e^{\sigma\xi(t)}}{\int e^{\sigma\xi(s)} ds} dt \\
&= \exp\Big\{\lambda(u_L) + O(y^2 z u_L^{-2}) + \omega(u_L) + \sigma u_L + \sigma w + \frac{\sigma y^2}{2\Delta u_L} + \frac{A\sigma u_L}{24}\gamma^4\Big\} \\
&\quad \times \frac{1}{\sqrt{\Delta\sigma u_L}} \exp\Big\{ - \frac{\sigma\Delta(u_L - z)}{2}\gamma^2 + \frac{z}{2u_L} - \frac{A}{24\Delta^2\sigma u_L} E\left(Z^4 | Z \le \zeta_L\right) \Big\} \\
&\quad \Big\{ E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big| \ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y \Big] \\
&\quad + E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 \\
&\qquad + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L} Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z) \ \Big| \ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y \Big] \Big\}
\end{aligned}$$

Using Taylor expansion on the two expectation terms, we obtain that

$$\begin{aligned}
& E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big| \ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y \Big] \\
&\quad + E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 \\
&\qquad + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L} Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z) \ \Big| \ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y \Big] \\
= \ & E\Big[p(x)(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z) \\
&\qquad - \frac{p'(x)}{2\sqrt{\sigma\Delta u_L}}(\gamma\sqrt{\sigma\Delta(u_L - z)} - Z)^2 \ \Big| \ Z \le \sqrt{1 - \frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L - z)}} y \Big] \\
& \times \exp\left\{ \frac{E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L} - Z)^3 + \frac{Ap(x)}{24\Delta^2\sigma^2 u_L} Z^4(\gamma\sqrt{\sigma\Delta u_L} - Z) \, | Z \le \zeta_L \Big]}{p(x) E(\gamma\sqrt{\sigma\Delta u_L} - Z | Z \le \zeta_L)} + o(u_L^{-1} + y u_L^{-1}) \right\}.
\end{aligned}$$

We insert the above identity back to the expression of $v'(x)$ and obtain that

$$
\begin{aligned}
v'(x) &= \exp\Big\{\lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\omega(u_L)+\sigma u_L+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\sigma u_L}{24}\gamma^4\Big\}\\
&\quad\times\frac{1}{\sqrt{\Delta\sigma u_L}}\exp\Big\{\frac{z}{2u_L}-\frac{A}{24\Delta^2\sigma u_L}E(Z^4|Z\le\zeta_L)\Big\}\\
&\quad\times H_{L,x}\Big(\gamma\sqrt{\sigma\Delta(u_L-z)},\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y;u_L\Big)\\
&\quad\times\exp\left\{\frac{E\Big[\frac{p''(x)}{6\sigma\Delta u_L}(\gamma\sqrt{\sigma\Delta u_L}-Z)^3+\frac{Ap(x)}{24\Delta^2\sigma^2u_L}Z^4(\gamma\sqrt{\sigma\Delta u_L}-Z)\,|Z\le\zeta_L\Big]}{p(x)E(\gamma\sqrt{\sigma\Delta u_L}-Z|Z\le\zeta_L)}\right\},
\end{aligned}
$$

where

$$H_{L,y}(x,\zeta;u)\triangleq e^{-\frac{x^2}{2}}\times E\Big[p(y)(x-Z)-\frac{p'(y)}{2\sqrt{\Delta\sigma u}}(x-Z)^2\ \Big|\ Z\le\zeta\Big].$$

■

Proof of Lemma 8. We insert $\gamma_L=\frac{\zeta_L}{\sqrt{\sigma\Delta u_L}}$ to the expression of $\mathcal{A}$ in (31) and obtain that

$$
\begin{aligned}
\mathcal{A} &= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\zeta_L^4}{24\Delta^2\sigma u_L}+\frac{z}{2u_L}-\frac{AE\left(Z^4|Z\le\zeta_L\right)}{24\Delta^2\sigma u_L}\\
&\quad+G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y;u_L\Big)-G_L(\zeta_L;u_L)\\
&\quad+\frac{E[\frac{p''(L)}{6\sigma\Delta u_L}(\zeta_L-Z)^3+\frac{Ap(L)}{24\Delta^2\sigma^2u_L}Z^4(\zeta_L-Z)\,|Z\le\zeta_L]}{p(L)E(\zeta_L-Z\,|Z\le\zeta_L)}.
\end{aligned}
$$

Note that $\Xi_L=-\lim_{u_L\to\infty}\partial_\zeta^2G_L(\zeta_L,u_L)$. Then,

$$
\begin{aligned}
\mathcal{A} &= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{A\zeta_L^4}{24\Delta^2\sigma u_L}+\frac{z}{2u_L}-\frac{AE\left(Z^4|Z\le\zeta_L\right)}{24\Delta^2\sigma u_L}\\
&\quad+\frac{E[\frac{p''(L)}{6\sigma\Delta u_L}(\zeta_L-Z)^3+\frac{Ap(L)}{24\Delta^2\sigma^2u_L}Z^4(\zeta_L-Z)\,|Z\le\zeta_L]}{p(L)E(\zeta_L-Z|Z\le\zeta_L)}\\
&\quad-\frac{\Xi_L+o(1)}{2}\Big(\frac{\zeta_Lz}{2u_L}+\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2.\\
&= \lambda(u_L)+o(yu_L^{-1})+O(y^2zu_L^{-2})+\sigma w+\frac{\sigma y^2}{2\Delta u_L}+\frac{z}{2u_L}+\frac{\kappa_L}{u_L}-\frac{\Xi_L+o(1)}{2}\Big(\frac{\zeta_Lz}{2u_L}+\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y\Big)^2.
\end{aligned}
$$

where κ_L is given as in (9). ■

Proof of Lemma 9. In this case that $|\sqrt{1-\frac{z}{u_L}}\zeta_L-\sqrt{\frac{\sigma}{\Delta(u_L-z)}}y-\zeta_L|>\varepsilon$, the maximum of $|v'(x)|$ is not necessarily attained at $x=L$. Note that this does not change the calculation very much except that the terms $p(x)$ and $p'(x)$ in $H_{x,L}$ may not be evaluated on the boundary $x=L$, but still in the region $[L-u^{-1/2+\delta}, L]$. Therefore, maximizing (29), we have that

$$
\begin{aligned}
& \sup_{x\in[L-u^{-1/2+\delta},L]} \log\left|H_{L,x}(\gamma\sqrt{\sigma\Delta(u_L-z)}, \sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y; u_L)\right| \\
= \ & G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y; u_L\Big) + O(u^{-1/2+\delta}).
\end{aligned}
$$

Therefore, we only need to add an $O(u^{-1/2+\delta})$ to the definition of $\mathcal{A}$ in (31). Furthermore, the term in (31) is bounded by

$$
G_L\Big(\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y, u_L\Big) - G_L(\zeta_L, u_L) \le -\delta_0\varepsilon^2
$$

for some $\delta_0 > 0$. Furthermore, on the set $\mathcal{L}_u^*$ we have that $\lambda(u_L) + o(yu_L^{-1}) + O(y^2zu_L^{-2}) = o(1)$. Therefore, we have the bound $S(w,y,z) \ge u_L^2 + w^2 + \frac{\Delta^2(w+z)^2}{A-\Delta^2} + 2u_L\mathcal{A}/\sigma + \delta_0\varepsilon^2 u_L$ and further

$$
P\left(\max_{x\in[L-u_L^{-1/2+\delta},L]} |v'(x)| > b; \mathcal{L}_{u_L}^*; \quad \left|\sqrt{1-\frac{z}{u_L}}\zeta_L - \sqrt{\frac{\sigma}{\Delta(u_L-z)}}y - \zeta_L\right| \ge \varepsilon\right) \quad = \quad o(1)u_L^{-1}e^{-u_L^2/2}.
$$

■

D Numerical Examples

In this section, we present one numerical example. We consider the differential equation in $[0, 2.5]$, that is, $L = 2.5$. The Gaussian process has zero mean and unit variance. The covariance function is

$$
C(t) = e^{-t^2/2}
$$

and thus $\xi(x)$ is infinitely differentiable. Furthermore, we consider a constant force $p(x) = 1$ and thus $F(x) = x$. We compute the tail probability $P(\max_{x\in[0,L]} |v'(x)| > b)$ via the approximations in the theorems, denoted by $\tilde{w}(b)$, and furthermore we compute the probabilities via importance sampling, denoted by $\hat{w}(b)$. For the Monte Carlo estimator, we choose the sample sizes such that the estimated standard deviations of the estimator is at the most 10% of $\hat{w}(b)$. Figure **??** shows the ratio between $\tilde{w}(b)/\hat{w}(b)$ as a function of $\log(b)$. The ratio stabilizes to one as b becomes large, but the convergence is quite slow as the smallest probability in Figure **??** is on the order of 10^{-9}.

We further consider a nonconstant force term $p(x) = \max(10 - 10(x-2.5)^2, 1)$ in the interval $[0, 5]$. The covariance function is $C(t) = e^{-0.3t^2}$. The corresponding plot of $\tilde{w}(b)/\hat{w}(b)$ versus $\log(b)$ is given by Figure **??**. The empirical rate of convergence of the non-constant case is much slower than that of the constant case.

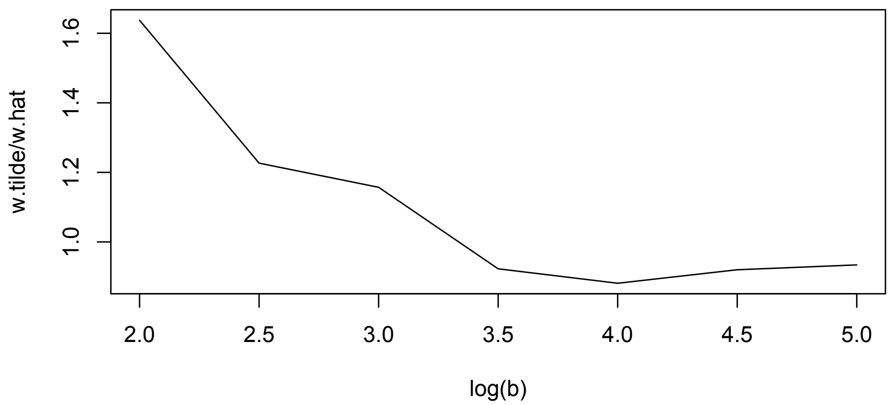

Figure 2: The ration $\tilde{w}(b)/\hat{w}(b)$ versus $\log(b)$ for $p(x) = 1$

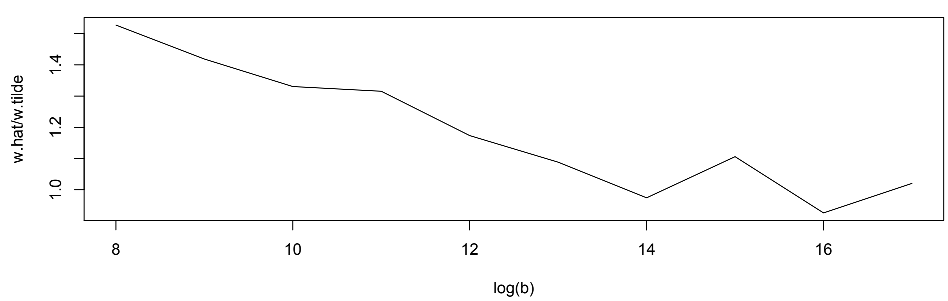

Figure 3: The ration $\hat{w}(b)/\tilde{w}(b)$ versus $\log(b)$ for non-constant $p(x)$.