Supplemental materials: technical development of positive recurrence and drift function for linear model

In this section, we construct a small set and drift function for the iterative Markov chain for the linear model in Section 5. Note that it suffices to consider the following sufficient statistics for block c,

$$\sum_{i\in c} x_i y_i, \quad \sum_{i\in c} x_i^2.$$

That is, we need to identify a small set in R^2 and a drift function to that small set for the two statistics in the above display. Given that we only consider one-step transition of the Markov process, we use "$\tilde{}$" to denote the updated values of the next iteration and (x_i, y_i)'s to denote the observed value or the imputed values from the previous iteration. Also, we adopt the following notation,

$$s^2_{\alpha,x} = \sum_{i\in\alpha} x_i^2, \quad s^2_{\alpha,y} = \sum_{i\in\alpha} y_i^2, \quad \tilde{s}^2_{\alpha,x} = \sum_{i\in\alpha} \tilde{x}_i^2, \quad \tilde{s}^2_{\alpha,y} = \sum_{i\in\alpha} \tilde{y}_i^2,$$

for $\alpha = a, b, c$. In what follows, we investigate the one step transition of $\sum_c x_i y_i$ and $\sum_c x_i^2$. To simplify the calculation and without loss of generality, we assume

$$\sum_a x_i y_i = 0.$$

Remark 12 *The construction of small set and drift function for the cases when $\sum_a x_i y_i \neq 0$ is completely analogous and more tedious. Also, we can perform a linear transformation on x or y and make the crossproduct equal to zero.*

Throughout this section, we adopt the following notations. Let n denote the sample size. We write a $a_n = O(b_n)$ if there exists $C > 0$ such that $a_n \le C b_n$; $a_n = o(b_n)$ if $\lim a_n/b_n = 0$. We write $x_n = O_2(a_n)$ if there exists a random variable $x > 0$ such that $|x_n|$ is stochastically dominated by $a_n x$ with $Ex^2 < \infty$ and $x_n = o_2(1)$ if $Ex_n^2 \to 0$.

The general strategy of constructing a small set and a drift function is to first identify an equilibrium point and let the small set C be a compact domain around the equilibrium point. For instance, $\sum_{i\in c} x_i y_i \approx 0$ and $\sum_{i\in c} x_i^2 \approx (s^2_{a,x} + s^2_{b,x})\frac{n_c}{n_a+n_b}$. Whence a small set has been identified, we are ready to construct the drift function. The basic idea is that if the current state of the Markov chain is far away from C, the chain will in expectation move closer to C. Therefore, we need to first compute approximations of

$$g(x_i; i\in c) \triangleq E(\sum_c \tilde{x}_i y_i | x_i; i\in c), \quad \text{and } f(x_i; i\in c) \triangleq E(\sum_c \tilde{x}_i^2 | x_i; i\in c).$$

The second step is to show that both g and f are contraction mappings with one unique fixed point. The small set C is then chosen to be a domain around this fixed point. In addition, we show that the noise compared with the drift is ignorable as long as the chain is far away enough from C. In Sections C.1 and C.2, we study the one-step transition of $\sum_{i\in c} x_i y_i$ and $\sum_{i\in c} x_i^2$. In Section C.3, we give the specific form of a drift function and small set C based on the results in Section C.1 and C.2. The calculations are the same for the Gibbs chain and the iterative chain. Therefore, we do not particularly differentiate them.

C.1 One-step transition of the cross-product

The iterative imputation evolves as such that we first impute the missing y in b and then impute the missing x in c. Therefore

$$\sum_b x_i \tilde{y}_i = \sum_b x_i(\beta_{y|x} x_i + \varepsilon_i) = \beta_{y|x} s_{b,x}^2 + \sum_b x_i \varepsilon_i,$$

where $\beta_{y|x}$ is a random variable following the posterior distribution given the observations in groups a and c and is asymptotically a normal random variable

$$N\left(\frac{\sum_a x_i y_i + \sum_c x_i y_i + O(1)}{s_{a,x}^2 + s_{c,x}^2 + O(1)}, \quad \frac{\tau_y^2 + O(1)}{s_{a,x}^2 + s_{c,x}^2 + O(1)}\right).$$

The term with $O(1)$ is the impact of the prior distribution and τ_y^2 is a random variable following the corresponding posterior distribution. In addition, ε_i's are i.i.d. $N(0, \tau_y^2)$. Therefore,

$$\begin{aligned}
\sum_b x_i \tilde{y}_i &= \frac{\sum_a x_i y_i + \sum_c x_i y_i + O(1)}{s_{a,x}^2 + s_{c,x}^2 + O(1)} s_{b,x}^2 + Z s_{b,x} \sqrt{1 + \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)}}, \\
&= \frac{\sum_a x_i y_i + \sum_c x_i y_i}{s_{a,x}^2 + s_{c,x}^2 + O(1)} s_{b,x}^2 + Z s_{b,x} \sqrt{1 + \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)}} \\
&\quad + \frac{O(1) s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)},
\end{aligned} \tag{36}$$

where $EZ = 0$ and $Z = O_2(\tau_y)$.

Similarly, conditional on the imputed y values in block b, the imputed x values in block c (for the next iteration) satisfies,

$$\begin{aligned}
\sum_c \tilde{x}_i y_i &= \sum_b y_i(\beta_{x|y} y_i + \varepsilon_i) \\
&= \frac{\sum_a x_i y_i + \sum_b x_i \tilde{y}_i}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} s_{c,y}^2 + Z' s_{c,y} \sqrt{1 + \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}} + \frac{O(1) s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}.
\end{aligned}$$

Plugging in (36) and $\sum_a x_i y_i = 0$ into the above display,

$$\begin{aligned}
\sum_c \tilde{x}_i y_i &= \sum_c x_i y_i \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} \\
&\quad + Z' s_{c,y} \sqrt{1 + \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}} + Z s_{b,x} \sqrt{1 + \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}} \\
&\quad + \frac{O(1) s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} + \frac{O(1) s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}.
\end{aligned}$$

where $E(Z) = E(Z') = 0$, $Z = O_2(\tau_y)$ and $Z' = O_2(\tau_x)$. The two terms in the last row of the above display with $O(1)$ are due to the prior. We write them as IP (impact of prior), that is

$$IP = \frac{O(1) s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} + \frac{O(1) s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)}.$$

Then, the above display can be simplified to

$$\sum_c \tilde{x}_i y_i = \sum_c x_i y_i \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} + O_2\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right) + IP.$$

We assume that for some $\varepsilon > 0$,

$$s_{b,x}^2 < (1 - 2\varepsilon) s_{a,x}^2, \quad s_{c,y}^2 < (1 - 2\varepsilon) s_{a,y}^2. \tag{37}$$

Remark 13 *The above assumption is strong. It requires the fraction of missing information to be small enough and is usually not necessary. This is just to simplify our analysis.*

Let

$$\gamma = \frac{s_{b,x}^2}{s_{a,x}^2 + s_{c,x}^2 + O(1)} \frac{s_{c,y}^2}{s_{a,y}^2 + \tilde{s}_{b,y}^2 + O(1)} \in (0, 1 - \varepsilon),$$

then

$$\begin{aligned}
\sum_c \tilde{x}_i y_i &= \gamma \sum_c x_i y_i + O_p\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right) + IP, \\
&= \gamma \sum_c x_i y_i + O_p\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right).
\end{aligned} \tag{38}$$

The last step is because IP(impact of prior) is of constant order $O(1)$. An intuitive interpretation of the above result is that $\sum_c x_i y_i$ decays exponentially fast to zero with rate γ.

C.2 One-step transition of the sum of squares

Now, we proceed to the one step transition of $s_{c,x}^2 = \sum_c x_i^2$. Let

$$\bar{\sigma}_x^2 = \frac{s_{a,x}^2 + s_{b,x}^2}{n_a + n_b}, \quad \bar{\sigma}_y^2 = \frac{s_{a,y}^2 + s_{c,y}^2}{n_a + n_c}.$$

Let $\rho_{a,c}$ be the sample correlation between x and y based on samples in a and c, and $\tilde{\rho}_{a,b}$ be that based on a and b samples. The sums of squares of the x and y's satisfy the following recursion,

$$\begin{aligned}\tilde{s}^2_{b,y} &= \rho^2_{a,c}\frac{s^2_{a,y}+s^2_{c,y}}{s^2_{a,x}+s^2_{c,x}}s^2_{b,x}+(1-\rho^2_{a,c})\bar{\sigma}^2_y n_b+O_2(\sqrt{n_b}) \\ &= \bar{\sigma}^2_y n_b\left[(1-\rho^2_{a,c})+\rho^2_{a,c}\frac{(n_a+n_c)s^2_{b,x}}{n_b(s^2_{a,x}+s^2_{c,x})}\right]+O_2(\sqrt{n_b}),\end{aligned} \tag{39}$$

Similarly,

$$\tilde{s}^2_{c,x}=\bar{\sigma}^2_x n_c\left[(1-\tilde{\rho}^2_{a,b})+\tilde{\rho}^2_{a,b}\frac{(n_a+n_b)s^2_{c,y}}{n_c(s^2_{a,y}+\tilde{s}^2_{b,y})}\right]+O_2(\sqrt{n_c}). \tag{40}$$

Therefore, by plugging (39) into (40), the evolution of $s^2_{c,x}$ satisfies

$$\frac{\tilde{s}^2_{c,x}}{\bar{\sigma}^2_x n_c}=(1-\tilde{\rho}^2_{a,b})+\tilde{\rho}^2_{a,b}\frac{s^2_{c,y}/n_c}{\frac{s^2_{a,y}}{n_a+n_b}+\frac{n_b\bar{\sigma}^2_y(1-\rho^2_{a,c})}{n_a+n_b}+\frac{\rho^2_{a,c}\bar{\sigma}^2_y s^2_{b,x}/[(n_a+n_b)\bar{\sigma}^2_x]}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\frac{s^2_{c,x}}{\bar{\sigma}^2_x n_c}}+O_2(\sqrt{n_b}/(n_a+n_b))}+O_2(1/\sqrt{n_c}).$$

Define function,

$$f(\lambda,\rho,\tilde{\rho})=(1-\tilde{\rho}^2)+\tilde{\rho}^2\frac{s^2_{c,y}/n_c}{\frac{s^2_{a,y}}{n_a+n_b}+\frac{n_b\bar{\sigma}^2_y(1-\rho^2)}{n_a+n_b}+\frac{\rho^2\bar{\sigma}^2_y s^2_{b,x}/[(n_a+n_b)\bar{\sigma}^2_x]}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}}.$$

Then, the evolution of $s_{c,x}$ follows,

$$\frac{\tilde{s}^2_{c,x}}{\bar{\sigma}^2_x n_c}=f\left(\frac{s^2_{c,x}}{\bar{\sigma}^2_x n_c},\rho_{a,c},\tilde{\rho}_{a,b}\right)+O_2(n_c^{-1/2}+n_b^{-1/2}) \tag{41}$$

Let λ^* be the solution to

$$f\left(\lambda^*,\rho_{a,c},\tilde{\rho}_{a,b}\right)=\lambda^*. \tag{42}$$

The expression λ^* depends on $\rho_{a,c}$ and $\tilde{\rho}_{a,b}$. To simplify notation, we omit the indexes of $\rho_{a,c}$ and $\rho_{a,b}$ in the notation of λ^*. In what follows, we provide conditions under which f is a contraction mapping with fixed point λ^*. Consider

$$\begin{aligned}\frac{\partial f(\lambda,\rho_{a,c},\tilde{\rho}_{a,b})}{\partial\lambda} &= \frac{\tilde{\rho}^2_{a,b}s^2_{c,y}/n_c}{\frac{s^2_{a,y}}{n_a+n_b}+\frac{n_b\bar{\sigma}^2_y(1-\rho^2_{a,c})}{n_a+n_b}+\frac{\rho^2_{a,c}\bar{\sigma}^2_y s^2_{b,x}/(n_a+n_b)\bar{\sigma}^2_x}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}} \\ & \quad \frac{\frac{\rho^2_{a,c}\bar{\sigma}^2_y s^2_{b,x}/(n_a+n_b)\bar{\sigma}^2_x}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}}{\frac{s^2_{a,y}}{n_a+n_b}+\frac{n_b\bar{\sigma}^2_y(1-\rho^2_{a,c})}{n_a+n_b}+\frac{\rho^2_{a,c}\bar{\sigma}^2_y s^2_{b,x}/(n_a+n_b)\bar{\sigma}^2_x}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}} \\ & \quad \frac{\frac{n_c}{n_a+n_c}}{\frac{s^2_{a,x}}{\bar{\sigma}^2_x(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}.\end{aligned}$$

The first term on the right hand side of the above display,

$$\frac{\tilde{\rho}_{a,b}^2 s_{c,y}^2/n_c}{\frac{s_{a,y}^2}{n_a+n_b}+\frac{n_b\bar{\sigma}_y^2(1-\rho_{a,c}^2)}{n_a+n_b}+\frac{\rho_{a,c}^2\bar{\sigma}_y^2 s_{b,x}^2/(n_a+n_b)\bar{\sigma}_x^2}{\frac{s_{a,x}^2}{\bar{\sigma}_x^2(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda}} \le \frac{s_{c,y}^2(n_a+n_b)}{s_{a,y}^2 n_c};$$

the second term is less than or equals to one; the last term,

$$\frac{\frac{n_c}{n_a+n_c}}{\frac{s_{a,x}^2}{\bar{\sigma}_x^2(n_a+n_c)}+\frac{n_c}{n_a+n_c}\lambda} \le \frac{n_c\bar{\sigma}_x^2}{s_{a,x}^2}.$$

We put all these terms together and obtain,

$$\frac{\partial f(\lambda,\rho_{a,c},\tilde{\rho}_{a,b})}{\partial\lambda} \le \frac{s_{c,y}^2(s_{a,x}^2+s_{b,x}^2)}{s_{a,y}^2 s_{a,x}^2}. \tag{43}$$

Suppose that, for some $\varepsilon > 0$,

$$\frac{s_{c,y}^2(s_{a,x}^2+s_{b,x}^2)}{s_{a,y}^2 s_{a,x}^2} < 1-\varepsilon. \tag{44}$$

Remark 14 *Similar to condition* (37)*, we assume* (44) *to simplify the complexity of the analysis. It is usually not necessary.*

We obtain that $|\partial_\lambda f(\lambda,\rho,\tilde{\rho})| < 1-\varepsilon$ for all $\lambda > 0$. One nice feature of having $|\partial_\lambda f(\lambda,\rho_{a,c},\tilde{\rho}_{a,b})| < 1-\varepsilon$ is that $f: R^+ \to R^+$ is a contraction mapping and for any $\Delta\lambda$,

$$|f(\lambda^*+\Delta\lambda,\rho_{a,c},\tilde{\rho}_{a,b})-\lambda^*| \le (1-\varepsilon)|\Delta\lambda|,$$

where $\lambda^* = f(\lambda^*,\rho,\tilde{\rho})$, uniqueness and existence of which have been proved in standard functional analysis. Therefore, with condition (44), $\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c}$ goes exponentially fast to λ^*.

C.3 The small set and drift function

Based on the fluid dynamics of $s_{c,x}^2$ and $\sum_c x_i y_i$, we are able to provide a drift function and a small set. Let $x_c = \{x_i : i \in c\}$ and λ^* be the solution to

$$f(\lambda^*,\rho_{a,c},E\tilde{\rho}_{a,b}) = \lambda^*.$$

Consider

$$V(x_c) = \frac{\left(\sum_c x_i y_i\right)^2}{s_{a,x}^2} + \frac{\left(\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c}-\lambda^*\right)^2}{A^2}n_c.$$

for some A large enough. Let $\tilde{\lambda}^*$ be the solution to

$$f(\tilde{\lambda}^*,\tilde{\rho}_{a,c},E(\tilde{\rho}_{a,b}|\tilde{x}_c)) = \tilde{\lambda}^*,$$

which is the equilibrium point of the next iteration, that is,

$$V(\tilde{x}_c) = \frac{\left(\sum_c \tilde{x}_i y_i\right)^2}{s_{a,x}^2} + \frac{\left(\frac{\tilde{s}_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \tilde{\lambda}^*\right)^2}{A^2} n_c.$$

Now we define a small set

$$C_A = \{x_c : V(x_c) \leq A\}.$$

In C_A both $\sum_c x_i y_i / s_{c,x}^2$ and $\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*$ are $1/\sqrt{n}$ distance away from zero. It is not hard to show that C_A is a small set. Consider the one step transition. Let $\zeta = 1 - \varepsilon$. For all $x_c \bar{\in} C_A$, thanks to (38), (41), (43), and $\gamma < \zeta$,

$$\begin{aligned}
V(\tilde{x}_c) &= \frac{\left(\sum_c \tilde{x}_i y_i\right)^2}{s_{a,x}^2} + \frac{\left(\frac{\tilde{s}_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \tilde{\lambda}^*\right)^2}{A^2} n_c \\
&\leq \zeta^2 \frac{\left(\sum_c x_i y_i\right)^2}{s_{a,x}^2} + O_2\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right) \frac{\sum_c x_i y_i}{s_{a,x}^2} + O_1\left(\frac{s_{c,y}^2 + s_{b,x}^2}{s_{a,x}^2}\right) \\
&\quad + \frac{n_c}{A^2}\left[\zeta^2\left(\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right)^2 + \left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| \left|\tilde{\lambda}^* - \lambda^*\right| + \left(\tilde{\lambda}^* - \lambda^*\right)^2\right] \\
&\quad + \frac{1}{A^2} O_2\left(\sqrt{n_c}\right)\left(\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| + \left|\tilde{\lambda}^* - \lambda^*\right| + O_2\left(\frac{1}{\sqrt{n_c}}\right)\right) \\
&\leq \zeta^2 V(x_c) \\
&\quad + O_2\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right) \frac{\sum_c x_i y_i}{s_{a,x}^2} + O_1\left(\frac{s_{c,y}^2 + s_{b,x}^2}{s_{a,x}^2}\right) \\
&\quad + \frac{n_c}{A^2}\left[\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| \left|\tilde{\lambda}^* - \lambda^*\right| + \left(\tilde{\lambda}^* - \lambda^*\right)^2\right] \\
&\quad + \frac{1}{A^2} O_2\left(\sqrt{n_c}\right)\left(\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| + \left|\tilde{\lambda}^* - \lambda^*\right| + O_2\left(\frac{1}{\sqrt{n_c}}\right)\right).
\end{aligned}$$

In what follows, we show that we can choose A sufficiently large, so that when $V(x_c) > A$

$$\begin{aligned}
&E\Big\{O_2\left(\sqrt{s_{c,y}^2 + s_{b,x}^2}\right) \frac{\sum_c x_i y_i}{s_{a,x}^2} + O_2\left(\frac{s_{c,y}^2 + s_{b,x}^2}{s_{a,x}^2}\right) \\
&+ \frac{n_c}{A^2}\left[\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| \left|\tilde{\lambda}^* - \lambda^*\right| + \left(\tilde{\lambda}^* - \lambda^*\right)^2\right] \\
&+ \frac{1}{A^2} O_2\left(\sqrt{n_c}\right)\left(\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2 n_c} - \lambda^*\right| + \left|\tilde{\lambda}^* - \lambda^*\right| + O_2\left(\frac{1}{\sqrt{n_c}}\right)\right)\Big\} \leq \varepsilon V(x_c)/2.
\end{aligned} \tag{45}$$

The first terms of the above display are all bounded by

$$\sqrt{V(x_c)}O_1\left(\sqrt{\frac{s_{c,y}^2+s_{b,x}^2}{s_{a,x}^2}}\right);$$

the second term

$$O_2\left(\frac{s_{c,y}^2+s_{b,x}^2}{s_{a,x}^2}\right)=O_2(1).$$

We focus on the second line in (45). Because λ^* is a smooth function of $\rho_{a,c}$, there exists from Taylor's expansion a κ such that

$$\left|\lambda^*-\tilde{\lambda}^*\right|\sqrt{n_c}\le\frac{\kappa\left|\sum_c x_iy_i-\sum_c\tilde{x}_iy_i\right|}{s_{a,x}^2}\sqrt{n_c}\le\frac{2\kappa\left|\sum_c x_iy_i\right|\sqrt{n_c}}{s_{a,x}^2}+O_2(1)\le 2\kappa\sqrt{\frac{n_c}{s_{a,c}^2}V(x_c)}$$

Thus,

$$\frac{n_c}{A^2}\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2n_c}-\lambda^*\right|\left|\tilde{\lambda}^*-\lambda^*\right|\le\frac{2\kappa}{A}\sqrt{\frac{n_c}{s_{a,c}^2}}V(x_c),\qquad\frac{n_c\left(\tilde{\lambda}^*-\lambda^*\right)^2}{A^2}\le\frac{4\kappa^2n_c}{A^2s_{a,c}^2}V(x_c),$$

and

$$\begin{aligned}&\frac{1}{A^2}O_2\left(\sqrt{n_c}\right)\left(\left|\frac{s_{c,x}^2}{\bar{\sigma}_x^2n_c}-\lambda^*\right|+\left|\tilde{\lambda}^*-\lambda^*\right|+O_2(\frac{1}{\sqrt{n_c}})\right)\\ \le\ &\frac{O_2(1)}{A^2}\left(A\sqrt{V(x_c)}+2\kappa\sqrt{\frac{n_c}{s_{a,c}^2}}\sqrt{V(x_c)}+O_2(1)\right).\end{aligned}$$

Therefore, for A sufficiently large and $V(x_c)>A$, (45) holds and

$$E(V(\tilde{x}_c))\le(1-\varepsilon/2)V(x_c).$$

Therefore, the Markov chain of the iterative imputation under conditions in (37) and (44) is positive recurrent and the expected recurrent time to the small set C_A is bounded by $V(x_c)+bI_{C_A}(x_c)$.